

ON PROPER MODULI SPACE OF SMOOTHABLE KÄHLER-EINSTEIN FANO VARIETIES

CHI LI, XIAOWEI WANG, AND CHENYANG XU

ABSTRACT. In this paper, we investigate the geometry of the orbit space of the closure of the subscheme parametrising smooth Fano Kähler-Einstein manifolds inside an appropriate Hilbert (or Chow) scheme. In particular, we prove that being K-semistable is a Zariski open condition and establish the uniqueness for the Gromov-Hausdorff limit for a punctured flat family of Fano Kähler-Einstein manifolds, which corresponds to a minimal orbit in a limiting orbit. Putting them together, we then construct a compact Hausdorff algebraic space parameterizing all K-polystable smoothable \mathbb{Q} -Fano varieties, and establish various properties which make it a good moduli space.

CONTENTS

1. Introduction	1
1.1. Main results	2
2. Preliminaries	4
3. Linear action of reductive groups on projective spaces	6
4. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano pair	8
4.1. Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds	9
4.2. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano family	9
5. Strong uniqueness for $0 < \beta \ll 1$	13
6. Continuity method	14
7. K-semistability of the nearby fibers	22
7.1. Orbit of K-semistable points	22
7.2. Zariski Openness of K-semistable varieties	23
7.3. Proof of Theorem 1.1 and 1.2	24
8. Local geometry near a smoothable K-polystable \mathbb{Q} -Fano variety	26
9. Appendix	32
9.1. Constructibility of kst	32
9.2. Stabilizer Preserving Property	34
References	37

1. INTRODUCTION

Constructing moduli spaces for higher dimensional algebraic varieties is a fundamental problem in algebraic geometry. For dimension one case, the moduli space parametrizing Deligne-Mumford stable curves was constructed via various kind of methods, e.g. geometric invariant theory (GIT), Teichmüller space quotient by mapping class group, etc. For higher dimensional case, one of the natural classes to consider is all canonically polarised manifolds, for which GIT machinery is quite successful (see [Aub78, Yau78, Vie95, Don01]). However, to construct a geometrically natural compactification for these moduli spaces, the GIT method in its classical form fails to produce that (cf. [WX14]), thus people have to develop substitutes. In fact, it has been quite a while for people to realize what kind of varieties should be included in order to form a *proper* moduli (cf. [KSB88]). Thanks to the recent breakthrough coming from the theory of minimal model program (see [BCHM10] etc.), one is able to obtain a rather satisfactory theory on proper projective moduli spaces parameterizing

Date: September 27, 2016.

KSBA-stable varieties, named after Kollár-Shepherd-Barron-Alexeev (see [Kol13] for a concise survey of this theory). We also remark that it is realized later that this compactification should coincide with the compactification from Kähler-Einstein metric/K-stability (cf. [Oda13, WX14, BG14]).

As for Fano varieties, the story is much subtler. Apart from some local properties, e.g. having only *Kawamata log terminal* (klt) singularities when a Fano variety is assumed to be K-semistable (cf. [Oda13]) and admitting klt Fano degenerations as long as a general fiber is a klt Fano variety in a one parameter family (cf. [LX14]), it is still not clear what kind of general Fano varieties we should parametrize in order for us to obtain a nicely behaved moduli space, especially if we aim to find a compact Hausdorff one, and how to construct it. The recent breakthrough in Kähler-Einstein problem, namely the solution to the Yau-Tian-Donaldson Conjecture ([CDS15a, CDS15b, CDS15c, Tia15]) is a major step forward, especially for understanding those Fano manifolds with Kähler-Einstein metrics. Furthermore, it implies that the right limits of smooth Kähler-Einstein manifolds form a bounded family. In this paper, we aim to use the analytic results they established to investigate the geometry of the compact space of orbits which is the closure of the space parametrizing smooth Fano varieties.

1.1. Main results. Our first main result of this paper is the following:

Theorem 1.1. *Let $\mathcal{X} \rightarrow C$ be a flat family of projective varieties over a pointed smooth curve $(C, 0)$ with $0 \in C$. Suppose*

- (1) $K_{\mathcal{X}}$ is \mathbb{Q} -Cartier and $-K_{\mathcal{X}/C}$ is relatively ample over C ;
- (2) for any $t \in C^\circ := C \setminus \{0\}$, \mathcal{X}_t is smooth and \mathcal{X}_0 is klt;
- (3) \mathcal{X}_0 is K-polystable.

Then

- (i) *there is a Zariski open neighborhood U of $0 \in C$ on which \mathcal{X}_t is K-semistable for all $t \in U$, and K-stable if we assume further \mathcal{X}_0 has a discrete automorphism group;*
- (ii) *for any other flat projective family $\mathcal{X}' \rightarrow C$ satisfying (1)-(3) as above and*

$$\mathcal{X}' \times_C C^\circ \cong \mathcal{X} \times_C C^\circ,$$

we can conclude $\mathcal{X}'_0 \cong \mathcal{X}_0$;

- (iii) *\mathcal{X}_0 admits a weak Kähler-Einstein metric. If we assume further that \mathcal{X}_t is K-polystable, then \mathcal{X}_0 is the Gromov-Hausdorff limit of \mathcal{X}_t endowed with the Kähler-Einstein metric for any $t \rightarrow 0$.*

If both \mathcal{X}_0 and \mathcal{X}'_0 are assumed to be smooth Kähler-Einstein manifolds then part of Theorem 1.1 is a consequence of the work [Sze10], where the more general case for arbitrary polarization is established. When the fiber is of dimension 2, this is also implied by the work of [Tia90, OSS16] as explicit compactifications of Kähler-Einstein Del Pezzo surfaces are constructed there.

Now let us give a brief account of our approach. First we note that although part of our theorem is stated in algebro-geometric terms, the proof indeed relies heavily on known analytic results, especially the recent work in [CDS15b, CDS15c, Tia15]. On the other hand, we remark that no further analytic tools are developed beyond their work in our paper. So our argument is actually more of an algebro-geometric nature.

The first main tool for us is a continuity method very similar to the one proposed by Donaldson in [Don12a]. Indeed, by throwing in an auxiliary divisor $\mathcal{D} \in |-mK_{\mathcal{X}}|$, we consider the following log extension of Theorem 1.1.

Theorem 1.2. *For a fixed $\beta \in [0, 1]$, let $\mathcal{X} \rightarrow C$ be a flat family over a pointed smooth curve $(C, 0)$ with a relative codimension 1 cycle \mathcal{D} over C . Suppose*

- (1) $-K_{\mathcal{X}/C}$ is ample and $\mathcal{D} \sim_C -mK_{\mathcal{X}/C}$ for some positive integer $m > 1$;
- (2) for any $t \in C^\circ := C \setminus \{0\}$, \mathcal{X}_t and \mathcal{D}_t are smooth, $(\mathcal{X}_0, \frac{1}{m}\mathcal{D}_0)$ is klt;
- (3) $(\mathcal{X}_0, \mathcal{D}_0)$ is β -K-polystable. (cf. Definition 2.3).

Then

- (i) *there is a Zariski neighborhood U of $0 \in C$, on which $(\mathcal{X}_t, \mathcal{D}_t)$ is β -K-semistable (in fact β -K-polystable if $\beta < 1$) for all $t \in U$;*

- (ii) for any other flat projective family $(\mathcal{X}', \mathcal{D}') \rightarrow C$ with a relative codimension 1 cycle \mathcal{D}' satisfying (1)-(3) as above and

$$(\mathcal{X}', \mathcal{D}') \times_C C^\circ \cong (\mathcal{X}, \mathcal{D}) \times_C C^\circ,$$

we can conclude $(\mathcal{X}'_0, \mathcal{D}'_0) \cong (\mathcal{X}_0, \mathcal{D}_0)$;

- (iii) $(\mathcal{X}_0, \mathcal{D}_0)$ admits a conical weak Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta)/m)$ along \mathcal{D}_0 , which is Gromov-Hausdorff limit of $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ endowed with the conical Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta_i)/m)$ along $\mathcal{D}_{t_i} \subset \mathcal{X}_{t_i}$ for any sequence $t_i \rightarrow 0$ and $\beta_i \nearrow \beta$.

To prove Theorem 1.2, one notices that the uniqueness is well-understood when the angle is small. We give an account to this fact using a completely algebro-geometric means. To be precise, we use the result that the set of log canonical thresholds satisfies ascending chain condition (ACC) (see [HMX14]) to show that when the angle β is smaller than a positive number $\beta_0 > 0$ there is only one extension with at worst klt singularities. Fix ϵ , such that $0 < \epsilon < \beta_0$. We define a set $\mathbf{B} \subset [\epsilon, 1]$ (cf. Section 6 for the precise definition) for which the conclusions of Theorem 1.2 hold for the angles belonging to the set \mathbf{B} . The result on small angle case implies $\mathbf{B} \supset [\epsilon, \beta_0]$.

Now to prove Theorem 1.1, let us first *assume* that all the nearby fibers \mathcal{X}_t are K-semistable. Then it suffices to show that \mathbf{B} is open and closed in $[\epsilon, 1]$. We establish them using two facts. First we prove a simple but very useful fact (see Lemma 3.1), which says that for a point p on the *limiting orbit* with *reductive stabilizer*, there is a Zariski open neighborhood $p \in U$ such that the closure of the $\mathrm{SL}(N+1)$ -orbit of any point in the limiting orbit near p actually contains $g \cdot p$ for some $g \in \mathrm{SL}(N+1)$. In particular, it guarantees that there is no nearby non-equivalent K-polystable points on the limiting orbit. With this, using a crucial *Intermediate Value Theorem* type of results (cf. Lemma 6.9), we show that if there is a different limit, which a priori could be far away from the given central fiber in the parametrizing Chow variety, then we can indeed always find another limit which either specializes to (X_0, D_0) in a test configuration or becomes the central fiber of a test configuration of (X_0, D_0) , violating the K-stability assumption. Similarly, this argument can also be applied to study the case when $\beta \nearrow 1$.

To finish the proof, we need to verify the assumption that all the nearby fibers \mathcal{X}_t are *K-semistable*. For this, one needs two observations. First, it follows from the work of [CDS15b, CDS15c, Tia15] that to check K-semistability of \mathcal{X}_t , $t \neq 0$, it suffices to test for all one-parameter-group (1-PS) degenerations in a fixed \mathbb{P}^N . Second, it follows from a straightforward GIT argument that *K-semistable threshold* (kst) (cf. Section 7.2) is a constructible function. So what remains to show is that it is also lower semi-continuous (also observed in [SSY14]), which is a consequence of the upper semi-continuity of the dimension of the automorphism groups and the continuity method deployed in the proof of Theorem 1.2.

With all this knowledge in hand, we will prove that there is a well-behaved orbit space for smoothable K-semistable Fano varieties.

Theorem 1.3. *For $N \gg 0$, let Z^* be the semi-normalization of the open set of $\mathrm{Chow}(\mathbb{P}^N)$ parametrizing all smoothable K-semistable Fano varieties in \mathbb{P}^N (see Section 8 for the precise definition of Z^*). Then the algebraic stack $[Z^*/\mathrm{SL}(N+1)]$ admits a proper good moduli space (see for the definition in [Alp13, Section 1.2]). Furthermore, for sufficiently large N , \mathcal{KF}_N does not depend on N .*

Therefore, our quotient is a compact Hausdorff Moishezon space thanks to [Art70, Theorem 7.3]. The existence of a moduli space for Kähler-Einstein Fano manifolds is well expected after the work of [Tia90]. A local quotient picture was suggested in [Don08, Section 5.3] and [Sze10], and was explicitly conjectured in [Spo12, Section 1.3 and 1.4] and [OSS16, Conjecture 6.2]. Furthermore, the moduli space is speculated to be projective by the existence of the descending of the CM-line bundle (see e.g. [PT06] and [OSS16]). We also remark that for smooth Kähler-Einstein Fano manifolds with discrete automorphism which are known to be asymptotically Chow stable by [Don01], they admit (possibly non-proper) algebraic moduli spaces thanks to the work of [Don15] and [Oda12].

Now let us say a few words of our approach to Theorem 1.3. Due to the lack of a GIT interpretation of the K-stability, a new strategy is needed to verify the existence of a *good* quotient moduli space. In order to apply the work of [AFS16] to obtain a good quotient, key ingredients that one needs to have include the stabilizer preserving condition for the local presentation of the moduli stack and

the affineness of the quotient morphism. We will establish both criterions by using a continuous slice lying over the stack, which is constructed via Tian's embedding of the Kähler-Einstein Fano varieties.

Finally we close the introduction by outlining the plan of the paper. In Section 2, we give the basic definitions. In Section 3, we review some facts on the linear action of a reductive group on a projective space. In Section 4, we list the main analytic results we need in this note. First we recall the recent results appeared in [CDS15b, CDS15c, Tia15]. Then we also state the Gromov-Hausdorff continuity for conical Kähler-Einstein metrics on a smooth family of Fano pairs (see [CDS15b, CDS15c, Tia15]). In Section 5, we prove that when the angle is small enough, the filling is always unique. In Section 6, we establish the main technical tool of our argument, which is a continuity theorem. We remark, with it we can already show Theorem 1.2 under the assumption that the nearby fibers are all β -K-polystable. In Section 7, we will prove the K-semi-stability of the nearby points by applying the continuity method. First in Section 7.1 we prove Theorem 7.2 which says that any orbit closure of a K-semistable Fano manifold contains only one isomorphic class of K-polystable \mathbb{Q} -Fano variety. In particular, this is an extension of the result of [CS14] for the Fano case. In Section 7.2, we show that a smoothing of a K-semistable \mathbb{Q} -Fano variety is always K-semistable. In Section 7.3, by putting all the results together, we finish the proof of Theorem 1.1 and 1.2. In Section 8, we apply our results and prove a Luna slice type theorem for K-stability, which is used to establish Theorem 1.3. In Section 9, we discuss two results which we need from the general theory of group action.

History. Now we remark on some history of this paper, which was original titled as ‘*Degeneration of Fano Kähler-Einstein manifolds*’ (see [LWX14]), and in the first version, we proved weaker results, e.g. the separateness of the moduli space. After it was posted on the arXiv, we were informed by the authors of [SSY14] who independently investigated similar questions with a circle of parallel ideas but in a more analytic fashion and obtained results which are closely related. In particular, in [SSY14], the authors obtained *first* the existence of weak Kähler-Einstein metrics on smoothable K-polystable \mathbb{Q} -Fano varieties; the *analytic* openness of K-stability in the case of *finite* automorphism group; the lower semi-continuity of the cone angle for conical Kähler-Einstein metrics. Those statements are not included in the first version of our preprint. As a consequence the uniqueness of K-stable filling with *finite* automorphism group was also obtained in [SSY14]. However, the approach in the first version of our paper naturally extends and give rise to a more complete picture as in the current version. We would like to thank the authors of [SSY14] for communicating their work to us. After we posted the second version of our paper on the arXiv, we were also informed by Odaka [Oda15] that he also has an independent construction of the moduli space of smoothable K-semistable Fano varieties as a proper algebraic space based on the results in [LWX14] and [SSY14].

Acknowledgments. The first author is partially supported by NSF: DMS-1405936. The second author is partially supported by a Collaboration Grants for Mathematicians from Simons Foundation and NSF:DMS-1609335, and he also wants to thank Professor D.H. Phong, Jacob Sturm and Jian Song for their constant encouragement over the years. The third author is partially supported by the grant ‘The Recruitment Program of Global Experts’. We are very grateful of Jacob Sturm for many valuable suggestions and comments. We also would like to thank Jarod Alper, Daniel Greb, Reyer Sjamaar and Chris Woodward for helpful comments. We are indebted to the anonymous referee for numerous useful suggestions. A large part of this work was done when CX visits the Institute for Advanced Study, which is partially sponsored by Ky Fan and Yu-Fen Fan Membership Funds, S.S. Chern Fundation and NSF: DMS-1128155, 1252158.

2. PRELIMINARIES

In this section, we will fix our convention of the paper. The definitions below are recalled from [Tia97] and [Don02]. The readers may also consult the lecture notes [PS10] and [Tho06] for both analytic and algebro-geometric point of view.

Definition 2.1. Let $(X, D; L)$ be an n -dimensional projective variety polarized by an ample line bundle L together with an effective divisor $D \subset X$. A *log test configuration* of $(X, D; L)$ consists of

- (1) A projective flat morphism $\pi : (\mathcal{X}, \mathcal{D}; \mathcal{L}) \rightarrow \mathbb{A}^1$;

- (2) A \mathbb{G}_m -action on $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, such that π is \mathbb{G}_m -equivariant with respect to the standard \mathbb{G}_m -action on \mathbb{A}^1 via multiplication;
(3) \mathcal{L} is relative ample and we have \mathbb{G}_m -equivariant isomorphism.

$$(1) \quad (\mathcal{X}^\circ, \mathcal{D}^\circ; \mathcal{L}|_{\mathcal{X}^\circ}) \cong (X \times \mathbb{G}_m, D \times \mathbb{G}_m; \pi_X^* L)$$

where $(\mathcal{X}^\circ, \mathcal{D}^\circ) = (\mathcal{X}, \mathcal{D}) \times_{\mathbb{A}^1} \mathbb{G}_m$ and $\pi_X : X \times \mathbb{G}_m \rightarrow X$.

A log test configuration is called a *product* test configuration if $(\mathcal{X}, \mathcal{D}; \mathcal{L}) \cong (X \times \mathbb{A}^1, D \times \mathbb{A}^1; \pi_X^* L)$ where $\pi_X : X \times \mathbb{A}^1 \rightarrow X$, and a *trivial* test configuration if $\pi : (\mathcal{X}, \mathcal{D}; \mathcal{L}) \rightarrow \mathbb{A}^1$ is a product test configuration with \mathbb{G}_m acting trivially on X .

In this article, we will focus on the projective pairs (X, D) satisfy the following

Definition 2.2. Let (X, D) be a projective pair with *Kawamata log terminal* (klt) singularities (see [KM98, 2.34]). Then (X, D) is said to be a *log Fano* pair if $-(K_X + D)$ is an ample \mathbb{Q} -Cartier divisor, and *\mathbb{Q} -Fano variety* if $D = 0$.

To proceed, let χ denote the Hilbert polynomial and we introduce $a_i, \tilde{a}_i, b_i, \tilde{b}_i \in \mathbb{Q}$ via the following expansions.

- $\chi(X, L^{\otimes k}) := \dim H^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2});$
- $\chi(D, (L|_D)^{\otimes k}) := \dim H^0(D, L^k|_D) = \tilde{a}_0 k^{n-1} + O(k^{n-2});$
- $w(k) := \text{weight of } \mathbb{G}_m\text{-action on } \wedge^{\text{top}} H^0(\mathcal{X}_0, \mathcal{L}^{\otimes k}|_{\mathcal{X}_0}) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1});$
- $\tilde{w}(k) := \text{weight of } \mathbb{G}_m\text{-action on } \wedge^{\text{top}} H^0(\mathcal{D}_0, \mathcal{L}^{\otimes k}|_{\mathcal{D}_0}) = \tilde{b}_0 k^n + O(k^{n-1}).$

Now we are ready to state the algebro-geometric criterion for the existence of conical Kähler-Einstein metric on a log Fano manifold (X, D) with cone angle $2\pi(1 - (1 - \beta)/m)$ along the divisor $D \in |-mK_X|$.

Definition 2.3. For a \mathbb{Q} -Fano variety X with $D \in |-mK_X|$ and a real number $\beta \in [0, 1]$, we define the *log generalized Futaki invariant with the angle β* as following:

$$\text{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = \text{DF}(\mathcal{X}; \mathcal{L}) + (1 - \beta) \cdot \text{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L})$$

with

$$\text{DF}(\mathcal{X}; \mathcal{L}) := \frac{a_1 b_0 - a_0 b_1}{a_0^2} \text{ and } \text{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L}) := \frac{1}{m} \cdot \frac{a_0 \tilde{b}_0 - b_0 \tilde{a}_0}{2a_0^2}.$$

Then

$$\text{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}^{\otimes r}) = \text{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}).$$

We say $(X, D; L)$ is called *β -K-semistable* if $\text{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, and *β -K-polystable* (resp. β -K-stable) if it is β -K-semistable with $\text{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ is a product test configuration (resp. trivial test configuration).

Thanks to the linear dependence of $\text{DF}_{1-\beta}(\mathcal{X}, \mathcal{D}; \mathcal{L})$ on β , we immediately obtain the following interpolation property:

Lemma 2.4. If $(X, D; L)$ is both β_1 -K-semistable and β_2 -K-polystable with $\beta_1 < \beta_2$ (resp. $\beta_2 < \beta_1$), then $(X, D; L)$ is β -K-polystable for any $\beta \in (\beta_1, \beta_2]$ (resp. $\beta \in [\beta_2, \beta_1)$).

Remark 2.5. Notice that if for $(X, D; K_X^{\otimes(-r)})$ where X is a \mathbb{Q} -Fano variety with $D \in |-mK_X|$,

$$\lambda : \mathbb{G}_m \rightarrow \text{SL}(N_r + 1) \text{ with } N_r + 1 := \dim H^0(X, K_X^{\otimes(-r)})$$

induces a test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$, then

$$(2) \quad \text{CH}(\mathcal{X}, \mathcal{D}; \mathcal{L}) = \frac{1}{mr^n} \cdot \left(\text{CH}(\mathcal{D}_0) - \frac{m}{r} \text{CH}(\mathcal{X}_0) \right)$$

with $\text{CH}(\mathcal{D}_0)$ and $\text{CH}(\mathcal{X}_0)$ being precisely the λ -weight for the Chow points of $\mathcal{D}_0, \mathcal{X}_0 \subset \mathbb{P}^{N_r}$.

3. LINEAR ACTION OF REDUCTIVE GROUPS ON PROJECTIVE SPACES

In this section, we prove a basic fact on a reductive group acting on \mathbb{P}^N , which will be crucial for the later argument. Let G be a reductive algebraic group acting on \mathbb{P}^N via a rational representation $\rho : G \rightarrow \mathrm{SL}(N+1)$ and $z : C \rightarrow \mathbb{P}^N$ be an algebraic morphism satisfying $z(0) = z_0 \in \mathbb{P}^N$ where $(0 \in C)$ is a smooth pointed curve germ. Let

$$\overline{BO} := \lim_{t \rightarrow 0} \overline{O_{z(t)}}$$

with $O_{z(t)} := G \cdot z(t)$ and $\overline{O_{z(t)}} \subset \mathbb{P}^N$ be its closure, that is, \overline{BO} is a union of (broken) orbits that $\overline{O_{z(t)}}$ specialized to.

Lemma 3.1. *Suppose $G_{z_0} < G$, the stabilizer of $z_0 \in \mathbb{P}^N$ for the G -action on \mathbb{P}^N , is reductive. Then there is a G -invariant Zariski open neighbourhood of $z_0 \in U \subset \mathbb{P}^N$ satisfying:*

$$(3) \quad \overline{O} \cap U = \bigcup_{\substack{O_p \subset \overline{O} \\ O_{z_0} \cap \overline{O_p} \neq \emptyset}} O_p \cap U \text{ where } O_p := G \cdot p \subset \overline{O},$$

i.e. the closure of the G -orbit of any point in \overline{O} near z_0 contains $g \cdot z_0$ for some (hence for all) $g \in G$. We will call O_{z_0} a minimal orbit.

Proof. We divide the proof into two steps:

Step 1: $G = G_{z_0}$. The representation $\rho : G \rightarrow \mathrm{SL}(N+1)$ induces a G -linearization of $\mathcal{O}_{\mathbb{P}^N}(1) \rightarrow \mathbb{P}^N$. Let $\chi : G \rightarrow \mathbb{G}_m$ be the character of the resulting G -action on $\mathcal{O}_{\mathbb{P}^N}(1)|_{z_0}$, since z_0 is fixed by G . Then z_0 is GIT poly-stable with respect to the linearization of $\mathcal{O}_{\mathbb{P}^N}(1)$ induced by the representation $\rho \otimes \chi^{-1} : G \rightarrow \mathrm{SL}(N+1)$. Our claim then follows from the standard fact of GIT that semi-stable points are Zariski open, i.e., there exists a G -invariant section s such that $s(z_0) \neq 0$ and we can take U the open set where s does not vanish. In particular, in this case U can be chosen as a G -invariant Zariski open set.

Step 2: $G > G_{z_0}$. Since G_{z_0} is reductive, we have a decomposition of its Lie algebra

$$\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_{z_0} \oplus \mathfrak{p}$$

as representations of G_{z_0} . The infinitesimal action of G at $0 \neq \hat{z}_0 \in \mathbb{C}^{N+1}$, a lifting of $z_0 \in \mathbb{P}^N$, induces a G_{z_0} -invariant decomposition $\mathbb{C}^{N+1} = \mathbb{C} \cdot \hat{z}_0 \oplus W' \oplus \mathfrak{p}$. By the proof of [Don12b, Proposition 1],

$$\mathbb{P}W = \mathbb{P}(W' \oplus \mathbb{C}\hat{z}_0) \subset \mathbb{P}^N$$

satisfies the following properties:

- (1) $z_0 \in \mathbb{P}W$ and is preserved by G_{z_0} ;
- (2) $\mathbb{P}W$ is transversal to the G -orbit of z_0 at z_0 ;
- (3) for $w \in \mathbb{P}W$ near z_0 and $\xi \in \mathfrak{g} := \mathrm{Lie}(G)$, if we let $\sigma_w : \mathfrak{g} \rightarrow T_w \mathbb{P}^N$ denote the infinitesimal action of G then

$$\sigma_w(\xi) \in T_w \mathbb{P}W \iff \xi \in \mathfrak{g}_{z_0} := \mathrm{Lie}(G_{z_0}).$$

In particular, part (3) implies that there exists a Zariski open neighborhood $U_0 \subset \mathbb{P}W$ of z_0 such that the infinitesimal action induced by \mathfrak{p}^\perp on $\mathbb{P}W$ is *transversal* for all points in U_0 (cf. Lemma 67).

Claim 3.2. Let $S := G \cdot \mathrm{Im}z \subset \mathbb{P}^N$ and H be the identity component of G_{z_0} . Then there is a Zariski open subset $U_W \subset U_0 \subset \mathbb{P}W$ and a collection of pointed arcs $\{z^i : (C_i, 0) \rightarrow (U_0, z_0)\}_{i=0}^d$ with $z^0 = z : C \rightarrow \mathbb{P}W$ such that

$$\overline{S} \cap U_W = \bigcup_{i=0}^d \overline{O(H, z^i)} \cap U_W \text{ with } O(H, z^i) := H \cdot \mathrm{Im}z^i \subset \mathbb{P}W.$$

Assume Claim 3.2 for the moment, let us define

$$\overline{BO}_i^W := \lim_{t \rightarrow 0} \overline{O_{z^i(t)}^W} \subset E \text{ with } O_{z^i(t)}^W = H \cdot z^i(t) \subset \overline{O(H, z^i)} \subset \mathbb{P}W$$

Next for each $0 \leq i \leq d$, applying *Step 1* to the H -action on $\mathbb{P}W$ and $\overline{BO_i^W} \subset \mathbb{P}W$, we obtain a H -invariant Zariski open $z_0 \in U'_i \subset \mathbb{P}W$ such that

$$\forall p \in U'_i \cap \overline{BO_i^W} \implies z_0 \in \overline{G \cdot p}.$$

Then $U = G \cdot \left(\bigcap_{i=0}^d U'_i \right)$ is the G -invariant Zariski open set we want.

Now let us proceed to the proof of Claim 3.2. To better illustrate the picture, let us treat the case $\dim G_{z_0} = 0$ first.

Case 1: $\dim G_{z_0} = 0$. Let us consider the variety $S := G \cdot \text{Im}z \subset \mathbb{P}^N$ and let $\partial S := \overline{S} \setminus S$. Then there is an open neighborhood $U_W \subset U_0$ such that $\overline{S} \cap U_W$ has only finitely many irreducible components. Let us write

$$\overline{S} \cap U_W = \bigcup_{i=0}^d C_i$$

with $C_0 = \text{Im}z(C)$ and C_i are irreducible components passing through z_0 .

Since $\partial \overline{S} \cap C_i$ is constructible, after a possible shrinking of C_i we have two possibilities:

- (1) $\partial \overline{S} \cap C_i = C_i$
- (2) $\partial \overline{S} \cap C_i = z_0$ or \emptyset .

We claim that the first case does not happen and then by choosing the arc $z^i : (C_i, 0) \rightarrow (U_0, z_0)$ we establish Claim 3.2. To prove our claim, one notices there are two kinds of possible points on $\partial \overline{S}$:

- *first kind:* a boundary point of $\overline{G \cdot z(t)}$ for a fixed t ;
- *second kind:* all the remaining points on $\partial \overline{S}$.

Notice that the set of both kinds of points form constructible sets. Any boundary point of the first kind can be indeed written as a limit of points in $G \cdot z(t) \cap U_W$ for a fixed t , but this is absurd as G acts on U_0 transversally. So we may assume all the points on C_i are of the *second kind*, this implies that

$$\text{Im}z \not\subset \overline{G \cdot z(t)} \text{ for a fixed } t \in C.$$

In particular, we have $\dim G + 1 = \dim \overline{S}$ as $\dim G_{z_0} = 0$. Since $\partial \overline{S}$ is G -invariant, we have $G \cdot C_i \subset \partial \overline{S}$. Now let us consider the G -action on $z \in C_i$, which implies that the

$$\dim \partial \overline{S} \geq \dim G + \dim C_i = \dim G + 1 = \dim \overline{S},$$

a contradiction. Thus our claim is verified.

Case 2: the general case. Let us consider the variety $S := G \cdot \text{Im}z \subset \mathbb{P}^N$ and let $\partial S := \overline{S} \setminus S$. Then there is an H -invariant open neighborhood $U_W \subset U_0$ such that $\overline{S} \cap U_W$ has only finitely many irreducible components, which are denoted by

$$\overline{S} \cap U_W = \bigcup_{i=0}^d V_i$$

with $V_0 = \overline{O(H, z)}$ and $z_0 \in V_i, 0 \leq i \leq d$. Moreover, V_i is H -invariant for each i since \overline{S} is.

Then Claim 3.2 amounts to saying that for each i , there is an arc $z^i : C_i \rightarrow U_0$ such that

$$V_i = \overline{O(H, z^i)} \cap U_W.$$

To find such an arc, all we need is a *general* $v \in V_i$ satisfying

$$(4) \quad \dim H \cdot v + 1 \geq \dim V_i,$$

since that implies two situations: either $\dim H \cdot v < \dim V_i$ for which we choose $z^i : C_i \rightarrow V_i$ be an arc joining z_0 and v so that $\text{Im}z^i \not\subset \overline{H \cdot v}$; or $\dim H \cdot v = \dim V_i$ for which we choose any nonconstant arc $z^i : C_i \rightarrow V_i$ satisfying $z^i(0) = z_0$. Then $\dim V_i = \dim O(H, z^i)$ and our Claim is justified.

To find such $v \in V_i$, we only need it to satisfy

$$\dim H \cdot v \geq \dim H \cdot z(t) \text{ for all } t \in C,$$

which again follows from the transversality. Indeed, there is a Zariski open set U_C of C , such that for any $t_0 \in U_C$,

$$\dim H \cdot z(t_0) = \max_{t \in C} \dim H \cdot z(t).$$

By definition of V_i , for a fixed general $v \in V_i$ there is a $g_i \in G$ and $t_0 \in U_C$ such that $g_i \cdot z(t_0) \in B(v, \epsilon) \in \mathbb{P}^N$, by the transversality of \mathfrak{p} -action on U_0 , for $\epsilon \ll 1$ there is an $h \in G$ close to identity such that $h \cdot g_i \cdot z(t_0) \in V_i$. By the genericity of v , we obtain

$$\dim H \cdot v \geq \dim H \cdot h \cdot g_i \cdot z(t_0) = \dim H \cdot z(t_0) \geq \dim H \cdot z(t) \text{ for all } t \in C.$$

and hence $\dim O(H, z^i) \geq \dim O(H, z)$ by our choice of $z^i : C_i \rightarrow V_i$.

Now we prove (4). Suppose (4) does not hold which is equivalent to $\dim V_i > \dim O(H, z^i)$, then we have

$$\begin{aligned} \dim \overline{S} &\geq \dim G \cdot V_i \\ (\mathfrak{p}\text{-acting on } U_0) &\geq \dim G/H + \dim V_i \\ &> \dim G/H + \dim O(H, z^i) \\ &\geq \dim G/H + \dim O(H, z) = \dim \overline{S}, \end{aligned}$$

a contradiction. So the proof of the Claim 3.2 and hence the Lemma are completed. \square

The necessity of the assumption that G_{z_0} is reductive can be illustrated by the following example.

Example 3.3. Let $M_2(\mathbb{C}) = \{[v, w] \mid v, w \in \mathbb{C}^2\}$ be the linear space of 2×2 matrices, on which $G := \text{GL}(2)$ is acting via multiplication on the left. Let $V := M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$, G acts on $\mathbb{P}V$ via the representation

$$\rho : \begin{array}{ccc} \text{GL}(2) & \longrightarrow & \text{SL}(V) \\ g & \longmapsto & \rho(g) \end{array} \quad \text{with } \rho(g) \cdot \begin{bmatrix} A \\ x_5 \\ x_6 \end{bmatrix} := \begin{bmatrix} g \cdot A \\ \det(g^{-1})x_5 \\ \det(g^{-1})x_6 \end{bmatrix}.$$

Let

$$z_0 = \begin{bmatrix} 0_{2 \times 2} \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad z'_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{P}V,$$

then their stabilizers are $G_{z_0} = G$ and $G_{z'_0} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} < \text{GL}(2)$. In particular, G_{z_0} is reductive while $G_{z'_0}$ is not. Now let

$$z(t) = \begin{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix} \\ 1 \\ t \end{bmatrix} \quad \text{and} \quad z'(t) = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \\ t \\ t^2 \end{bmatrix} \in \mathbb{P}V$$

be two curves in $\mathbb{P}V$, then we have

$$\lim_{t \rightarrow 0} \overline{O_{z(t)}} = \lim_{t \rightarrow 0} \mathbb{P}V_{[1, t]} = \lim_{t \rightarrow 0} \overline{O_{z'(t)}} = \mathbb{P}V_{[1, 0]},$$

where $V_{[1, t]} := \{tx_5 = x_6\} \subset V$. Clearly, $z_0 := z(0)$ satisfies (3) while $z'_0 := z'(0)$ does not, since

$$z'_0 \notin \mathbb{P}^1 \cong \overline{G \cdot z''_\epsilon} \subset \mathbb{P}V_{[1, 0]} \text{ for } 0 < |\epsilon| \ll 1 \text{ where } z''_\epsilon := \begin{bmatrix} 1 & \epsilon \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. GROMOV-HAUSDORFF CONTINUITY OF CONICAL KÄHLER-EINSTEIN METRIC ON SMOOTH FANO PAIR

In this section, we list the important analytic results that will be needed in our main argument.

4.1. Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds. In this subsection, let us recall the main technical results obtained in the solution of Yau-Tian-Donaldson conjecture (see [CDS15b, CDS15c, Tia15] and [Ber12]).

Theorem 4.1. *Let X_i be a sequence of n -dimensional Fano manifolds with a fixed Hilbert polynomial χ and $D_i \subset X_i$ be smooth divisors in $|-mK_{X_i}|$ for a fixed $m > 0$. Let $\beta_i \in (0, 1)$ be a sequence converging to β_∞ with $0 < \epsilon_0 \leq \beta_\infty \leq 1$. Suppose that each X_i admits Kähler metric $\omega_i(\beta_i)$ solving:*

$$(5) \quad \text{Ric}(\omega(\beta_i)) = \beta_i \omega(\beta_i) + \frac{1 - \beta_i}{m} [D] \text{ on } X_i .$$

that is, $\omega_i(\beta_i)$ is a conical Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta_i)/m)$ along the divisor $D_i \subset X_i$. Then the Gromov-Hausdorff limit of any subsequence of $\{(X_i, \omega_i(\beta_i))\}_i$ is homeomorphic to a \mathbb{Q} -Fano variety Y . Furthermore, there is a unique Weil divisor $E \subset Y$ such that

- (1) $(Y, \frac{1 - \beta_\infty}{m} E)$ is klt;
- (2) Y admits a weak conical Kähler-Einstein metric solving

$$\text{Ric}(\omega(\beta_\infty)) = \beta_\infty \omega(\beta_\infty) + \frac{1 - \beta_\infty}{m} [E] \text{ on } Y .$$

In particular, $\text{Aut}(Y, E)$ is reductive and the pair (Y, E) is β_∞ -K-polystable;

- (3) *possibly after passing to a subsequence, there are embeddings $T_i : X_i \rightarrow \mathbb{P}^N$ and $T_\infty : Y \rightarrow \mathbb{P}^N$, defined by the complete linear system $|-rK_{X_i}|$ and $|-rK_Y|$ respectively for $r = r(m, \epsilon_0, \chi)$ and $N + 1 = \chi(X_i, K_{X_i}^{-\otimes r})$, such that $T_i(X_i)$ converge to $T_\infty(Y)$ as projective varieties and $T_i(D_i)$ converge to $T_\infty(E)$ as algebraic cycles.*

In the following corollary, we denote by $C^{\alpha, \beta}$ the space of conical Kähler metrics defined in [Don12a].

Corollary 4.2. *Let (X, D) be smooth Fano pair with $D \in |-mK_X|$. Then*

- (1) *(X, D) is β -K-stable if and only if it admits a conical Kähler-Einstein metric $\omega(\beta) \in C^{\alpha, \beta}$ solving (5).*
- (2) *Let $\gamma \in (0, 1]$. Then (X, D) is γ -K-semistable if and only if it admits a conical Kähler-Einstein metric $\omega(\beta) \in C^{\alpha, \beta}$ solving (5) for any $\beta \in (0, \gamma)$.*

Remark 4.3. Notice that the limiting divisor $E \subset Y$ is actually \mathbb{Q} -Cartier. To see that, one notice that on the smooth locus of Y

$$(6) \quad E|_{Y^{\text{reg}}} \sim -mK_{Y^{\text{reg}}},$$

which implies $E|_Y \sim -mK_Y$ as Y is normal. On the other hand, Y being \mathbb{Q} -Fano implies that K_Y is \mathbb{Q} -Cartier. This together with (6) implies that E is \mathbb{Q} -Cartier.

4.2. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano family.

Definition 4.4. Let us introduce

$$(7) \quad \mathbb{P}^{d, n; N} := \mathbb{P}(\text{Sym}^d(\mathbb{C}^{N+1})^{\otimes(n+1)}) .$$

and for any degree d and n -dimensional algebraic cycle $X \subset \mathbb{P}^N$, let $\text{Chow}(X) \in \mathbb{P}^{d, n; N}$ denote its Chow point.

To set the scene, let

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{D}) & \xrightarrow{i} & \mathbb{P}^N \times \mathbb{P}^N \times \Delta \\ \downarrow \pi & & \downarrow \\ \Delta & \xlongequal{\quad} & \Delta \end{array}$$

be projective flat family of Fano varieties over the disc $\Delta = \{|t| < 1\} \subset \mathbb{C}$ such that:

- (1) \mathcal{X} is smooth and $\mathcal{D} \in |-mK_{\mathcal{X}/\Delta}|$ is a smooth divisor defined by a smooth section $s_{\mathcal{D}} \in \Gamma(\Delta, \omega_{\mathcal{X}/\Delta}^{\otimes -m})$;
- (2) π is a holomorphic submersion (which is guaranteed if $(\mathcal{X}_t, \mathcal{D}_t)$ is smooth for all $t \in \Delta$ and the family $(\mathcal{X}, \mathcal{D}) \xrightarrow{\pi} \Delta$ is flat).

To get rid of the $U(N+1)$ -ambiguity for the later argument, let us assume that $\omega_{\mathcal{X}}^{\otimes -r}$ is relatively very ample and i be the embedding induced by a *prefixed basis*

$$\{s_i(t)\}_{i=0}^N \subset \Gamma(\Delta, \pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta}))$$

then $i^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta})$. Now let $(r\omega_{\text{FS}}(t), h_{\text{FS}}^{\otimes r}(t))$ denote the metric on $(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta})|_{\mathcal{X}_t})$ induced from the embedding i via the basis $\{s_i\}$. Suppose that for each $t \in \Delta$, \mathcal{X}_t is K-semistable. Then by Lemma 2.4, $(\mathcal{X}_t, \mathcal{D}_t)$ is β -K-polystable for any $\beta \in (0, 1)$. So by Corollary 4.2, for any $\beta \in (0, 1)$ there exists conical Kähler-Einstein metric $\omega(t, \beta)$ on the pair $(\mathcal{X}_t, \frac{1-\beta}{m}\mathcal{D}_t)$ which satisfies

$$\text{Ric}(\omega(t, \beta)) = \beta\omega(t, \beta) + \frac{1-\beta}{m}[\mathcal{D}_t].$$

In the following, by abusing of name, sometime we will abbreviate $\omega(t, \beta)$ as a conical Kähler-Einstein metric with *cone angle* β (instead of $2\pi(1 - (1-\beta)/m)$) *along* D , since the integer m is fixed once for all for the whole paper. Now assume $\omega(t, \beta) = \omega_{\text{KE}}(t, \beta) = \omega_{\text{FS}}(t) + \sqrt{-1}\partial\bar{\partial}\varphi(t, \beta)$ where $r \cdot \omega_{\text{FS}}(t)$ is equal to the Fubini-Study metric induced from the embedding of $\mathcal{X}_t \rightarrow \mathbb{P}^N$ using the basis $\{s_i(t)\}_{i=0}^N$. Then $\varphi(t, \beta)$ is the unique solution (c.f.[Ber15, Theorem 7.3]) to the equation

$$(8) \quad (\omega_{\text{FS}}(t) + \sqrt{-1}\partial\bar{\partial}\varphi(t, \beta))^n = e^{f(t) - \beta\varphi(t, \beta)} \frac{\omega_{\text{FS}}^n(t)}{\left(|s_{\mathcal{D}_t}|_{h_{\text{FS}}^{\otimes m}(t)}^2\right)^{\frac{1-\beta}{m}}},$$

where $f(t)$ satisfies

$$(9) \quad \text{Ric}(\omega_{\text{FS}}(t)) = \omega_{\text{FS}}(t) + \sqrt{-1}\partial\bar{\partial}f(t) \text{ and } \int_{\mathcal{X}_t} e^{f(t)} \cdot \omega_{\text{FS}}^n(t) = \int_{\mathcal{X}_t} \omega_{\text{FS}}^n(t).$$

Remark 4.5. It's easy to check that an equivalent form of equation (8) is:

$$(10) \quad (\omega_{\text{FS}}(t) + \sqrt{-1}\partial\bar{\partial}\varphi(t, \beta))^n \cdot |s_{\mathcal{D}_t}|_{h_{\text{FS}}^{\otimes m}e^{-\varphi}}(s_{\mathcal{D}_t} \otimes \overline{s_{\mathcal{D}_t}})^{\frac{1}{m}} = 1.$$

We define a positive definite Hermitian matrix

$$A_{\text{KE}}(t, \beta) = [(s_i, s_j)_{\text{KE}, \beta}(t)]$$

with

$$(s_i, s_j)_{\text{KE}, \beta}(t) = \int_{\mathcal{X}_t} \langle s_i(t), s_j(t) \rangle_{h_{\text{KE}}^{\otimes r}(t, \beta)} \omega^n(t, \beta),$$

where $h_{\text{KE}}(t, \beta) := h_{\text{FS}}(t) \cdot e^{-\varphi(t, \beta)}$. Now we introduce *r-th Tian's embedding*

$$(11) \quad T : (\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta)) \longrightarrow \mathbb{P}^N$$

to be the one given by the basis $\{g(t, \beta) \circ s_j(t)\}_{j=0}^N$ with $g(t, \beta) = A_{\text{KE}}^{-1/2}(t, \beta)$.

Definition 4.6. We denote by

$$(12) \quad \text{Chow}(\mathcal{X}_t, (1-\beta)\mathcal{D}_t) \in \mathbb{P}^{d, n; N} := \mathbb{P}^{d, n; N} \times \mathbb{P}^{\delta, n-1; N}$$

the Chow point of the pair $(\mathcal{X}_t, \mathcal{D}_t) \subset \mathbb{P}^N$ using Tian's embedding for the basis $\{s_i\}$ with respect to Kähler form $\omega(t, \beta)$, where (d, δ) are the degrees of $X \subset \mathbb{P}^N$ and $D \subset \mathbb{P}^N$ respectively. We note that when $\beta = 1$, the second factor $\mathbb{P}^{\delta, n-1; N}$ is not trivial as we still remember \mathcal{D}_t , i.e., $\text{Chow}(X, 0 \cdot D)$ is not the same as $\text{Chow}(X)$. See Remark 4.7.1 below.

Remark 4.7. We make some remarks:

- (1) It is by definition that

$$\text{Chow}(\mathcal{X}_t, (1-\beta)\mathcal{D}_t) = (\text{Chow}(\mathcal{X}_t), \text{Chow}(\mathcal{D}_t); \omega(t, \beta)).$$

In the following we will always use the coefficient $(1-\beta)$ to stress that the cycle is obtained via Tian's embedding with respect to the metric $\omega(t, \beta)$.

- (2) Tian's embedding is well defined for any klt \mathbb{Q} -Fano log pair with weak conical Kähler-Einstein metric $(X, (1-\beta)D; \omega_{\text{KE}}(\beta))$. Note that for any weak conical Kähler-Einstein metric $\omega_{\text{KE}}(\beta)$, we always assume that the local potential is bounded (see [BBE⁺11]).
- (3) The advantage of fixing a basis $\{s_i(t)\}_{i=0}^N \subset \Gamma(\Delta, \pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta}))$ lies in the fact that, the image of Tian's embedding and hence the Chow point, $\text{Chow}(\mathcal{X}_t, (1-\beta)\mathcal{D}_t)$ is completely determined by the isometric class of $\omega(t, \beta)$. See Lemma 4.9.

Proposition 4.8. $\text{Chow}(\mathcal{X}_t, (1-\beta)\mathcal{D}_t)$ varies continuously in $\mathbb{P}^{d,n;N}$ with respect to the pair $(\beta, t) \in (0, 1) \times \Delta$.

Proof. Using the above notations, we claim that $\varphi_{\text{KE}}(t, \beta)$ is continuous with respect to t for any $\beta < 1$. Assuming the claim, $A_{\text{KE}}(t, \beta)$ is then continuous with respect to t , and hence the images of Tian's embedding given by orthonormal basis change continuously.

Now we verify the claim by applying implicit function theorem. First we notice that the complex manifold $(\mathcal{X}_t, \mathcal{D}_t)$ is diffeomorphic to a fixed pair (X, D) endowed with the integrable complex structure J_t thanks to the assumption that π is a submersion. Let $C^{2,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t)$ and $C^{\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t)$ denote the function spaces on $(\mathcal{X}_t, \mathcal{D}_t; J_t)$ defined in [Don12a]. For each fixed $t \in \Delta$, we consider the map:

$$(13) \quad \begin{aligned} F(t, \beta, \cdot) : C^{2,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t) &\longrightarrow C^{\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t) \\ \varphi &\longmapsto \log \frac{(\omega_t + \sqrt{-1}\partial_{J_t}\bar{\partial}_{J_t}\varphi)^n|_{s_{\mathcal{D}_t}}|_{h_t}^{2(1-\beta)/m}}{\omega_t^n} - f_t + \beta\varphi \end{aligned}$$

where for simplicity we write $f_t = f(t)$, $\omega_t = \omega_{\text{FS}}(t)$ and $h_t = h_{\text{FS}}^{\otimes m}(t)$, and $s_{\mathcal{D}_t}$ is the defining section for \mathcal{D}_t as before. Note that $\varphi_{\text{KE}}(t, \beta)$ is exactly the solution to the equation $F(t, \beta, \varphi) = 0$. We would like to apply implicit function theorem to obtain the continuity of $\varphi_{\text{KE}}(t, \beta)$ with respect to t . In order to do that, we need to work with a *fixed* function space, whereas the spaces $C^{2,\alpha;\beta}(\mathcal{X}_t, \mathcal{D}_t; J_t)$ depends on the parameter t . To get around this, we notice that the metrics $\{\omega_t(\cdot, J_t \cdot)\}_t$ change smoothly and hence $C^{\alpha;\beta}(X, D; J_t) = C^{\alpha;\beta}(X, D; J_0)$. This key observation allows us to identify the space $C^{2,\alpha;\beta}(X, D; J_0)$ and $C^{2,\alpha;\beta}(X, D; J_t)$ via the following simple way. Let us fix a family of background conical Kähler metrics:

$$\hat{\omega}_t = \omega_t + \epsilon\sqrt{-1}\partial_{J_t}\bar{\partial}_{J_t}|_{s_{\mathcal{D}_t}}|_{h_t}^{2\gamma},$$

with $\gamma = 1 - \frac{1-\beta}{m} \in (0, 1)$ being fixed and $0 < \epsilon \ll 1$. Then we define a linear map:

$$(14) \quad \begin{aligned} Q_{t,\beta} := Q(t, \beta, \cdot) : C^{2,\alpha;\beta}(X, D; J_0) &\longrightarrow C^{2,\alpha;\beta}(X, D; J_t) \\ \tilde{\varphi} &\longmapsto (-\Delta_{\hat{\omega}_t} + 1)^{-1} \circ (-\Delta_{\omega_0} + 1)\tilde{\varphi}. \end{aligned}$$

Since $\ker(-\Delta_{\hat{\omega}_t} + 1) = \{0\}$ by the proof of [Don12a, Proposition 8], it follows from Donaldson's Schauder estimate in [Don12a, Section 4.3] that $Q_{t,\beta'}$ is an isomorphism for $|t| \ll 1$ and $\beta' \in (\beta - \epsilon, \beta + \epsilon)$ with $0 < \epsilon \ll 1$. Also using the explicit parametrix constructed in [Don12a, Section 3], Q_t gives rise to a *continuous* local linear trivialization of the family of subspaces $C^{2,\alpha;\beta}(X, D; J_t) \subset C^{\alpha;\beta}(X, D; J_t) = C^{\alpha;\beta}(X, D; J_0)$. Denoting $\tilde{\varphi}(t, \beta) = Q_{t,\beta}^{-1}(\varphi(t, \beta))$, we can calculate:

$$\left. \frac{\partial F(t, \beta, Q(t, \beta, \tilde{\varphi}))}{\partial \tilde{\varphi}} \right|_{(0, \beta, \tilde{\varphi}_{\text{KE}})}(\phi) = (\Delta_{\omega_{\text{KE}}} + \beta) \circ Q_0 \phi = (\Delta_{\omega_{\text{KE}}} + \beta) \phi$$

which is invertible by [Don12a, Theorem 2] since there exists no holomorphic vector field on the pair $(\mathcal{X}_0, \mathcal{D}_0)$ (see [SW12, Theorem 2.1] or Lemma 5.4). Now we can apply effective implicit function theorem as in [Don12a, Section 4.4] to the map $F(t, \beta, Q(t, \beta, \cdot)) : C^{2,\alpha;\beta}(X, D; J_0) \rightarrow C^{\alpha;\beta}(X, D; J_0)$ to get a continuous family of solutions $\tilde{\varphi}_{\text{KE}}(t, \beta')$ to the equation $F(t, \beta', Q(t, \beta', \tilde{\varphi})) = 0$ for $|t| \ll 1$ and $\beta' \in (\beta - \epsilon, \beta + \epsilon)$ with $0 < \epsilon \ll 1$. Since the argument for this last statement is standard, we will only sketch its proof. For a fixed β by the usual implicit function theorem we first get a family of solutions $\tilde{\varphi}_{\text{KE}}^{(1)}(t, \beta)$ to the equation $F(t, \beta, Q(t, \beta, \tilde{\varphi}_{\text{KE}}^{(1)})) = 0$ for $|t| \ll 1$. Then we can apply Donaldson's argument of deforming cone angles in [Don12a, Section 4.4] in a family version to further get $\tilde{\varphi}_{\text{KE}}(t, \beta')$ for any $|t| \ll 1$ and $\beta' \in (\beta - \epsilon, \beta + \epsilon)$. More precisely, let $\omega_{\text{KE}}(t, \beta) = \omega_{\text{FS}} + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_{\text{KE}}^{(1)}(t, \beta)$ be the continuous family of $C^{\alpha;\beta}$ conical Kähler-Einstein metric obtained earlier. For each $\beta' \in (\beta - \epsilon, \beta + \epsilon)$ and t near 0, we define the new reference metric $\omega(t, \beta') := \omega_{\text{KE}}(t, \beta) + \sqrt{-1}\partial\bar{\partial}(\|s_{\mathcal{D}_t}\|^{2\beta'/m} - \|s_{\mathcal{D}_t}\|^{2\beta/m})$ where $\|\cdot\|^2$ is a smooth extension of Hermitian metric determined by $h_{\text{FS}} \exp(-\tilde{\varphi}_{\text{KE}}^{(1)}(t, \beta))$ on $-mK_{\mathcal{X}_t}|_{\mathcal{D}_t}$ (using the fact that $\tilde{\varphi}_{\text{KE}}^{(1)}(t, \beta)$ is “smooth in tangential directions” by [Don12a, Section 4.3]). Then as in the proof of [Don12a, Proposition 7], one can show that

- (1) $k_{\beta'} := |s_{\mathcal{D}}|_{\beta'}^{-2\beta'/m} (s_{\mathcal{D}} \otimes \overline{s_{\mathcal{D}}})^{1/m} \omega(t, \beta')^n$ (see (10)) satisfies $\|k_{\beta'} - 1\|_{C^{\alpha;\beta'}} \rightarrow 0$ as $\beta' \rightarrow \beta$. Here we used $|\cdot|_{\beta'}^2$ to denote the Hermitian metric on $-mK_{\mathcal{X}_t}$ whose curvature is equal to $m \cdot \omega(t, \beta')$.

- (2) If $\Delta_{\beta'}$ denotes the Laplace operator associated to $\omega(t, \beta')$, then $\Delta_{\beta'} + \beta'$ is invertible and the operator norm of its inverse is bounded by a fixed constant independent of β' and t near 0.

So the effective version of implicit function theorem allows us to get a continuous family of solutions $\tilde{\varphi}_{\text{KE}}(t, \beta')$. In fact, notice that $\omega(t, \beta') = \omega_{\text{FS}} + \sqrt{-1}\partial\bar{\partial}\psi(t, \beta')$ where $\psi(t, \beta') = \tilde{\varphi}_{\text{KE}}^{(1)}(t, \beta) + \|s_{\mathcal{D}_t}\|^{2\beta'/m} - \|s_{\mathcal{D}_t}\|^{2\beta/m}$ is an “almost solution” of the conical Kähler-Einstein equation by the item (1) above and is continuous with respect to both t and β' . By item (2) and the effective implicit function theorem, we then know that the difference between the actual solution $\tilde{\varphi}_{\text{KE}}(t, \beta')$ and $\psi(t, \beta')$ approaches 0 in $C^{2,\alpha;\beta'}$ norm as $\beta' \rightarrow \beta$. As a consequence, $\tilde{\varphi}_{\text{KE}}(t, \beta')$ is continuous at $\beta' = \beta$ in C^0 -norm with respect to both β' and t . Noting that the argument above does not depend on the origin 0 and β we choose, hence $\varphi_{\text{KE}}(t, \beta') = Q(t, \beta', \tilde{\varphi}_{\text{KE}}(t, \beta'))$ is continuous with respect to all $t \in \Delta$ and $\beta' \in (\beta - \epsilon, \beta + \epsilon)$.

By using the complex Monge-Ampère equation in (8) or (10), we see that the family of volume forms $\omega_{\text{KE}}(t, \beta')^n$ on the fixed smooth manifold X is continuous in $L^p(X)$, $\forall p \in [1, 1/(1 - \beta'))$ with respect to β' and t . So the family of matrices of L^2 -inner products $A_{\text{KE}}(t, \beta') = [(s_i, s_j)_{\text{KE}, \beta'}(t)]$ is also continuous with respect to t and β' . So Tian’s embeddings $T(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta'))$ determined by $\{A_{\text{KE}}^{-1/2}(t, \beta') \circ s_j(t)\}_{j=0}^N$ indeed produce a continuous family of Chow forms inside $\mathbb{P}^{\mathbf{d}, n; N}$. \square

Let $\{(X_i, D_i)\}$ be a sequence of smooth Fano pairs with a fixed Hilbert polynomial χ and $D_i \in |-mK_{X_i}|$. Suppose each X_i ’s admit a unique conical Kähler-Einstein form $\omega(i, \beta_i)$ solving

$$\text{Ric}(\omega(i, \beta_i)) = \beta_i \omega(i, \beta_i) + \frac{1 - \beta_i}{m} [D_i] \text{ on } X_i$$

with $\inf \beta_i \geq \epsilon > 0$, we define

$$T_i : (X_i, D_i; \omega(i, \beta_i)) \longrightarrow \mathbb{P}^N$$

to be the Tian’s embedding with respect to $\omega(i, \beta_i)$ for sufficiently large N depending only on ϵ , m and the fixed Hilbert polynomial χ , and let $\text{Chow}(X_i, (1 - \beta_i)D_i) \in \mathbb{P}^{\mathbf{d}, n; N} \times \mathbb{P}^{\delta, n-1; N}$ denote the Chow point corresponding to the Tian’s embedding of X_i with respect to $\omega(i, \beta_i)$. Then we have

Lemma 4.9. *Let $(X, D) \subset \mathbb{P}^N$ be a log \mathbb{Q} -Fano pair with the same Hilbert polynomial χ and $D \in |-mK_X|$. Suppose (X, D) admits a weak conical Kähler-Einstein form $\omega(\beta)$ with $\beta = \lim_{i \rightarrow \infty} \beta_i$ solving*

$$\text{Ric}(\omega(\beta)) = \beta \omega(\beta) + \frac{1 - \beta}{m} [D] \text{ on } X.$$

Then

$$(X_i, D_i; \omega(i, \beta_i)) \xrightarrow{\text{GH}} (X, D; \omega'(\beta)) \text{ as } i \rightarrow \infty$$

for a conical Kähler-Einstein metric $\omega'(\beta)$ is equivalent to the following statement: there is a sequence of $\{g_i\} \subset \text{U}(N+1)$ such that

$$g_i \cdot \text{Chow}(X_i, (1 - \beta_i)D_i) \longrightarrow \text{Chow}(X, (1 - \beta)D) \in \mathbb{P}^{\mathbf{d}, n; N} \text{ as } i \rightarrow \infty,$$

where $\text{Chow}(X, (1 - \beta)D)$ denote the Chow point of Tian’s embedding $T : (X, D; \omega(\beta)) \rightarrow \mathbb{P}^N$ for a fixed basis $\{s_i\}$.

Proof. This follows directly from Theorem 4.1 which is in turn from the works of [CDS15b, CDS15c] and [Tia12, Tia15]. Indeed let us assume that $(X_i, D_i; \omega(i, \beta_i)) \xrightarrow{\text{GH}} (X, D; \omega'(\beta))$ where we can assume the limit exists by Theorem 4.1. Then by Theorem 4.1.(3) $(T_i(X_i), T_i(D_i)) \rightarrow (T'_\infty(X), T'_\infty(D))$ where T_i (resp. T'_∞) is given by Tian’s embedding determined by an orthonormal basis of $H^0(X_i, -mK_{X_i})$ (resp. $H^0(X, -mK_X)$) with respect to $\omega(i, \beta_i)$ (resp. $\omega'(\beta)$). Assume $\omega(\beta)$ is also a conical Kähler-Einstein metric on (X, D) . By the uniqueness of conical Kähler-Einstein metrics proved in [BBE⁺11], there exists a holomorphic automorphism $\sigma \in \text{Aut}(X, D)$ such that $\sigma^* \omega(\beta) = \omega'(\beta)$. Moreover because σ lifts to $\text{Aut}(X, D, -mK_X)$, there is a unitary isomorphism between $(H^0(X, -mK_X), \|\cdot\|_{\omega(\beta)}^2)$ and $(H^0(X, -mK_X), \|\cdot\|_{\omega'(\beta)}^2)$ where $\|\cdot\|_{\omega(\beta)}^2$ ($\|\cdot\|_{\omega'(\beta)}^2$) is the L^2 inner product induced by $\omega(\beta)$ (resp. $\omega'(\beta)$). Via this isomorphism, we have $(T_i(X_i), T_i(D_i)) \rightarrow (T_\infty(X), T_\infty(D))$ where T_∞ is given by Tian’s embedding determined by an orthonormal basis of $H^0(X, -mK_X)$ with respect to

$\omega(\beta)$. Now the statement of the lemma holds because the orthonormal basis of a unitary vector space is defined only up to $U(N+1)$ ambiguity. \square

5. STRONG UNIQUENESS FOR $0 < \beta \ll 1$

In this section, we will give a completely algebraic proof of the fact that when the angle $\beta > 0$ is sufficiently small, then there is a unique filling.

Proposition 5.1. *For a fixed a finite set $I \subset [0, 1]$, there exists a number $\beta_I > 0$ such that if $(X, (1 - \beta_I)D)$ is a klt pair, D is \mathbb{R} -Cartier and the coefficients of \mathbb{R} -divisor D are contained in I , then (X, D) is log canonical.*

Proof. By [HMX14, Theorem 1.1], we know that for the set of all n -dimensional log pairs (X, D) satisfying the property that D is a \mathbb{R} -divisor and its coefficients are contained in I , the set of log canonical thresholds

$$\{\text{lct}(X, D) \mid X \text{ is } n \text{ dimensional, the coefficients of } D \text{ are in } I\}$$

satisfies the *ascending chain condition* (ACC). In particular, there exists a maximum β_I among all log canonical thresholds which are strictly less than 1.

Then we know that if $(X, (1 - \beta_I)D)$ is klt and D is \mathbb{Q} -Cartier, (X, D) is log canonical, since otherwise, we will have a pair whose log canonical threshold is in $(1 - \beta_I, 1)$, which is a contradiction. \square

Let $\mathcal{X} \rightarrow C$ be a flat family of \mathbb{Q} -Fano varieties over a smooth pointed curve $0 \in C$. We assume \mathcal{X} is \mathbb{Q} -Gorenstein. Fixe $m > 1$ and $\mathcal{D} \sim_C -mK_{\mathcal{X}}$ is a divisor such that for *every* $t \in C$, the fiber $(\mathcal{X}_t, \frac{1}{m}\mathcal{D}_t)$ is klt. For instance, we can choose m sufficiently divisible such that $|-mK_{\mathcal{X}}|$ is relatively base point free over C and $\mathcal{D} \sim_C -mK_{\mathcal{X}}$ to be a divisor in the general position. In particular, \mathcal{D}_t is smooth for $t \in C^\circ$.

Theorem 5.2. *Let*

$$\beta_0 := \min \left\{ \beta_I, \frac{1}{m+1} \right\}$$

with β_I being given in Proposition 5.1 for the set $I = \{\frac{q}{m} \mid q = 1, 2, \dots, m\}$. For any fixed $\beta \in (0, \beta_0]$, suppose $(\mathcal{X}', \mathcal{D}') \rightarrow C$ is another flat family with $K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}'$ being \mathbb{Q} -Cartier and satisfies

$$(15) \quad (\mathcal{X}', \mathcal{D}') \times_C C^\circ \cong (\mathcal{X}, \mathcal{D}) \times_C C^\circ$$

and $(Y_\beta, \frac{1-\beta}{m}E_\beta) := (\mathcal{X}'_0, \frac{1-\beta}{m}\mathcal{D}'_0)$ being irreducible and klt. Then the above isomorphism can be extended to an isomorphism

$$(\mathcal{X}', \mathcal{D}') \cong (\mathcal{X}, \mathcal{D}).$$

Proof. Since $(\mathcal{X}_t, \mathcal{D}_t)$ is smooth for $t \in C^\circ$, the coefficients of $\frac{1}{m}E_\beta$ lies in the set $\{\frac{q}{m} \mid q \in \mathbb{N}\}$. By our assumption that $(Y_\beta, \frac{1-\beta}{m}E_\beta)$ is klt and $\beta \leq \beta_0 \leq \frac{1}{m+1}$, we have

$$\frac{1}{m+1} \leq \frac{1-\beta}{m} \text{ and } \frac{1-\beta}{m}c_i < 1 \text{ hence } c_i < m+1$$

where $E_\beta = \sum_i c_i E_{\beta,i}$ with $E_{\beta,i}$ being a prime divisor for each i . Hence the coefficients of $\frac{1}{m}E_\beta$ must lie in $I = \{\frac{q}{m} \mid q = 1, 2, \dots, m\}$. By our assumption of $\beta \in (0, \beta_0] \subset (0, \beta_I]$, we know that $(Y_\beta, \frac{1}{m}E_\beta)$ is log canonical by Proposition 5.1. Furthermore, since Y_β is irreducible, we know that

$$K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}' \sim_{\mathbb{Q}, C} 0$$

as this holds over C° .

Let W be a common resolution

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ \mathcal{X} & \xrightarrow{\phi} & \mathcal{X}' \end{array}$$

that is an isomorphism over C° . If the birational morphism ϕ extends to a birational morphism $\mathcal{X}_0 \xrightarrow{\phi|_{\mathcal{X}_0}} Y_\beta$, then

$$q^* K_{\mathcal{X}'} \sim_{C, \mathbb{Q}} p^* K_{\mathcal{X}}$$

as ϕ is an isomorphism in codimension one, which implies

$$\mathcal{X} = \text{Proj} \bigoplus_{r=0}^{\infty} \mathcal{O}_W(-rp^* K_{\mathcal{X}/C}) = \text{Proj} \bigoplus_{r=0}^{\infty} \mathcal{O}_W(-rq^* K_{\mathcal{X}'/C}) = \mathcal{X}'$$

and we are done already. So from now on we assume $\mathcal{X}_0 \not\sim_{\mathbb{Q}} Y_\beta$.

Now let us write

$$(16) \quad p^*(K_{\mathcal{X}} + \frac{1}{m}\mathcal{D}) + a_0 Y_\beta + \sum a_i E_i = K_W + \frac{1}{m} p_*^{-1} \mathcal{D}.$$

Since $(\mathcal{X}_0, \frac{1}{m}\mathcal{D}_0)$ is klt, this implies that $(\mathcal{X}, \frac{1}{m}\mathcal{D} + \mathcal{X}_0)$ is plt near \mathcal{X}_0 by inversion of adjunction [KM98, Theorem 5.50]. Hence for any divisor F whose center is contained in \mathcal{X}_0 we have

$$-1 < a(F, \mathcal{X}, \frac{1}{m}\mathcal{D} + \mathcal{X}_0) = a(F, \mathcal{X}, \frac{1}{m}\mathcal{D}) - v_F(\mathcal{X}_0) \leq a(F, \mathcal{X}, \frac{1}{m}\mathcal{D}) - 1$$

i.e. $(\mathcal{X}, \frac{1}{m}\mathcal{D})$ is terminal along \mathcal{X}_0 . Therefore, $a_0 > 0$ and $a_i > 0$. Similarly,

$$(17) \quad q^*(K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}') + b_0 \mathcal{X}_0 + \sum b_i E_i = K_W + \frac{1}{m} q_*^{-1} \mathcal{D}',$$

and we have $b_0, b_i \geq 0$ because $(Y, \frac{1}{m}E)$ is log canonical thanks to Proposition 5.1 and our choice of β . Since the right hand sides of (16) and (17) are equal to each other by (15), $\mathcal{X}_0 \not\sim_{\mathbb{Q}} Y_\beta$, and both $K_{\mathcal{X}} + \frac{1}{m}\mathcal{D}$ and $K_{\mathcal{X}'} + \frac{1}{m}\mathcal{D}'$ are \mathbb{Q} -linearly equivalent to a relatively *trivial* divisor over C , these imply there is a constant $c \leq 0$ such that

$$a_0 Y_\beta + \sum a_i E_i = b_0 \mathcal{X}_0 + \sum b_i E_i + c \cdot W_0.$$

By comparing the coefficients of Y_β on both sides, we see $c > 0$; but by comparing the coefficients of \mathcal{X}_0 on both sides, we see $c \leq 0$. This contradiction implies that $\mathcal{X}' = \mathcal{X}$. \square

Remark 5.3. If $m = 1$, the pair we get is plt instead of klt. The above argument indeed also applies to this case.

A similar uniqueness statement is observed in [Oda12, 4.3] and the above argument indeed gives a straightforward proof of it.

We also notice that the automorphism group $\text{Aut}(X, D)$ is always finite by the following well known fact.

Lemma 5.4. *Let (X, D) be a klt pair such that $-K_X$ is ample and $D \sim_{\mathbb{Q}} -K_X$. Then $\text{Aut}(X, D)$ is finite.*

Proof. We can choose a sufficiently small $\epsilon > 0$ such that $(X, (1 + \epsilon)D)$ is klt and we know $K_X + (1 + \epsilon)D$ is ample. As $\text{Aut}(X, D)$ preserves $K_X + (1 + \epsilon)D$, so it gives polarized automorphisms. Therefore, to prove it is finite, we only need to show that it does not contain \mathbb{G}_m or \mathbb{G}_a as a subgroup. For \mathbb{G}_m this follows from [HX11, Lemma 3.4]. As mentioned there, the same argument also works for \mathbb{G}_a verbatimly. \square

6. CONTINUITY METHOD

In this section, we will develop our continuity method which serves as the main technique of the proof of the main result. Let $(C, 0)$ be a smooth pointed curve, we define $C^\circ := C \setminus \{0\}$ as before. To begin with, let us fix $\mathfrak{B} \in (0, 1]$ and we will assume the nearby smooth fibers are all \mathfrak{B} -*K-polystable* for the rest of this section. We fix an $\epsilon \in (0, \beta_0)$, with β_0 being given as in Theorem 5.2. By Lemma 2.4, for any $\beta \in [\epsilon, \mathfrak{B}]$, $(\mathcal{X}_t, \mathcal{D}_t)$ is β -K-polystable. Applying [CDS15a, CDS15b, CDS15c, Tia12] (cf. Corollary 4.2), we conclude that $(\mathcal{X}_t, \mathcal{D}_t)$ admits a (unique when $\beta < 1$ thanks to [Ber15, Theorem 7.3]) conical Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta)/m)$ along \mathcal{D}_t for all $t \in C^\circ$ near 0. This leads us to introduce the following notion.

Definition 6.1. We say

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{D}; \mathcal{L}) & \longrightarrow & (\mathbb{P}\mathcal{E}; \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \\ \downarrow \pi & & \downarrow \\ C & \xlongequal{\quad} & C \end{array}$$

is a *Kähler-Einstein degeneration of index (r, \mathfrak{B})* if for any $\beta \in [\epsilon, \mathfrak{B}]$

- (1) $\mathcal{D} \in |-mK_{\mathcal{X}}|$;
- (2) $\mathcal{L} = K_{\mathcal{X}}^{\otimes -r}$ is relatively *very ample* and $\mathcal{E} = \pi_*\mathcal{L}$ is locally free of rank $N+1$;
- (3) $\forall t \in C$, $(\mathcal{X}_t, \frac{1}{m}\mathcal{D}_t)$ is klt and $(\mathcal{X}_t, \mathcal{D}_t)$ is a *smooth* Fano pair for $\forall t \in C^\circ$;
- (4) For $\beta < 1$ and $\forall t \in C^\circ$, $(\mathcal{X}_t, \mathcal{D}_t)$ admits a unique Kähler form $\omega(t, \beta) \in C^{\alpha, \beta}$ in the sense of [Don12a] solving

$$(18) \quad \text{Ric}(\omega(t, \beta)) = \beta\omega(t, \beta) + \frac{1-\beta}{m}[\mathcal{D}_t] \text{ on } \mathcal{X}_t.$$

Moreover, $\omega(t, \beta)$ gives rise to r -th Tian's embedding

$$T : (\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta)) \longrightarrow \mathbb{P}^N.$$

By Theorem 4.1, there is a uniform $r = r(\mathcal{X}, \mathcal{D})$ being independent of $\beta \in [\epsilon, \mathfrak{B}]$ such that all Gromov-Hausdorff limits of subsequences of the family $\{(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta))\}_{t \in C, \beta \in [\epsilon, \mathfrak{B}]}$ can be embedded in to \mathbb{P}^N .

Definition 6.2. Let us continue with the notation as above and define

$$\mathbf{B}_r(\mathcal{X}, \mathcal{D}) := \left\{ \beta \in [\epsilon, \mathfrak{B}] \left| \begin{array}{l} (X, D) \text{ admits a conical Kähler-Einstein metric } \omega(\beta) \text{ solving} \\ \text{Ric}(\omega(\beta)) = \beta\omega(\beta) + \frac{1-\beta}{m}[D] \text{ on } X. \\ \text{Moreover, } (\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta)) \xrightarrow{\text{GH}} (X, D; \omega(\beta)) \text{ as } t \rightarrow 0. \end{array} \right. \right\}$$

and we fix \mathbf{T} such that $\epsilon \leq \mathbf{T} \leq \sup\{\sigma \in [\epsilon, \mathfrak{B}] \mid [\epsilon, \sigma] \subset \mathbf{B}_r(\mathcal{X}, \mathcal{D})\}$.

By Theorem 4.1, the Gromov-Hausdorff limit of any subsequence of $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}, \omega(t_i, \beta))$ is a \mathbb{Q} -Fano Y together with a \mathbb{Q} -Cartier divisor E such that $(Y, \frac{1-\beta}{m}E)$ is log Fano.

Lemma 6.3. $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) \supset [\epsilon, \beta_0]$.

Proof. After shrinking C if necessary, we may choose a holomorphic basis

$$\{s_i(t)\}_{i=0}^N \subset \Gamma(\Delta, \pi_*\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\Delta}))$$

for the family $\mathcal{X} \rightarrow C$ as in Section 4.2, which gives rise to an *algebraic* arc

$$(19) \quad \begin{array}{ccc} z : C & \longrightarrow & \mathbb{P}^{\mathbf{d}, n; N} \times C \\ t & \longmapsto & (\text{Chow}(\mathcal{X}_t, \mathcal{D}_t), t). \end{array}$$

For this arc, we know that $\text{Chow}(Y, E)$ for the Gromov-Hausdorff limit $(Y, E; \omega_Y)$ of any subsequence $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega_{\text{KE}}(t_i, \beta))\}_{t_i \rightarrow 0}$ lies in the fiber over $0 \in C$ of the morphism

$$\begin{array}{ccc} \overline{\text{SL}(N+1) \cdot \text{Im}z} & \longrightarrow & \mathbb{P}^{\mathbf{d}, n; N} \times C \\ \downarrow \pi_C & & \downarrow \\ C & \xlongequal{\quad} & C. \end{array}$$

By choosing an arc $\tilde{z} : \tilde{C} \rightarrow \overline{\text{SL}(N+1) \cdot \text{Im}z}$ that passes through $\text{Chow}(Y, E)$ and dominates C , and comparing the universal family over $\text{Im}\tilde{z} \subset \mathbb{P}^{\mathbf{d}, n; N} \times C$ with the pull-back family induced by the map $\pi_C \circ \tilde{z} : \tilde{C} \rightarrow C$, we conclude that $(Y, E) = (X, D)$ as long as $\beta \leq \beta_0$ thanks to Theorem 5.2. Our proof is thus completed. \square

Remark 6.4. Notice that Lemma 6.3 implies that for $\beta \in [0, \beta_0]$, (X, D) is actually β -K-stable (see Lemma 5.4), which can also be proved by using Theorem 5.2 and a verbatim extension of the theory of special test configuration developed in [LX14] to the log setting. In fact, using the latter approach, we can indeed conclude a pair (X_0, D_0) is β -K-stable if $D_0 \sim -mK_{X_0}$, $(X_0, \frac{1}{m}D_0)$ is klt and $\beta \in [0, \beta_0]$, *without* assuming X_0 is smoothable. However, this stronger fact is not needed for the rest of the paper.

From now on, let us assume $(\mathcal{X}_0, \mathcal{D}_0)$ is \mathfrak{B} -K-polystable, we are going to show that $\mathbf{B}_r(\mathcal{X}, \mathcal{D})$ is both open and closed in the set $[\epsilon, \mathfrak{B}]$, or equivalently we can choose

$$\mathbf{T} = \mathfrak{B} = \max_{[\epsilon, \sigma] \subset \mathbf{B}_r(\mathcal{X}, \mathcal{D})} \{\sigma\}.$$

To do this, we first define a map

$$(20) \quad \begin{array}{ccc} \tau : [\epsilon, \mathfrak{B}] \times C^\circ & \longrightarrow & \mathbb{P}^{\mathbf{d}, n; N} \\ (\beta, t) & \longmapsto & \text{Chow}(\mathcal{X}_t, (1 - \beta)\mathcal{D}_t) \end{array} \quad (\text{cf. see Definition 4.6})$$

Then we have

Lemma 6.5. $\tau|_{[\epsilon, \mathfrak{B}] \times C^\circ}$ is continuous.

Proof. By Proposition 4.8, $\tau(\cdot, \cdot)$ is continuous with respect to (β, t) on $[\epsilon, \mathfrak{B}] \times C^\circ$. By Theorem 4.1, the Gromov-Hausdorff limit of $(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta_i))$ for any sequence $\beta_i \nearrow \mathfrak{B}$ is \mathfrak{B} -K-polystable and lies in $\overline{\text{SL}(N+1) \cdot \mathcal{X}_t}$. On the other hand, since $(\mathcal{X}_t, \mathcal{D}_t)$ is \mathfrak{B} -K-polystable, this implies the limit must lie in $\text{U}(N+1) \cdot \text{Chow}(\mathcal{X}_t, (1 - \mathfrak{B})\mathcal{D}_t)$, hence the metrics $\{h_{\text{KE}}(t, \beta)\}_{(t, \beta) \in [\epsilon, \mathfrak{B}] \times C^\circ}$ (cf. Section 4.2) vary continuously for $(\beta, t) \in [\epsilon, \mathfrak{B}] \times C^\circ$. So $\tau(\cdot, t)$ is also continuous at $\beta = \mathfrak{B}$ with respect to the basis $\{s_i\}$ in Definition 4.6. Thus the proof is completed. \square

By Lemma 6.3, we know that the continuity of $q \circ \tau$ can be extended to $[\epsilon, \beta_0] \times \{0\}$, where $q: \mathbb{P}^{\mathbf{d}, n; N} \rightarrow \mathbb{P}^{\mathbf{d}, n; N}/\text{U}(N+1)$ is the natural quotient morphism, which is continuous with respect to the quotient topology on $\mathbb{P}^{\mathbf{d}, n; N}/\text{U}(N+1)$. Next we will show indeed β -continuity of $q \circ \tau$ can be extended to $[\epsilon, \mathbf{T}] \times \{0\}$ (i.e. including the central fiber) as long as $q \circ \tau$ can be continuously extended to $[\epsilon, \mathbf{T}] \times C$ based on the fact that (X, D) is a degeneration of smooth pairs $(\mathcal{X}_t, \mathcal{D}_t)$ admitting conical Kähler-Einstein metrics $\omega(t, \beta)$ for any $\beta \in [\epsilon, \mathbf{T}]$. To do that, let us prefix a continuous distance function on $\mathbb{P}^{\mathbf{d}, n; N}$

$$(21) \quad \text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}} : \mathbb{P}^{\mathbf{d}, n; N} \times \mathbb{P}^{\mathbf{d}, n; N} \longrightarrow \mathbb{R}_{\geq 0}.$$

Lemma 6.6. Let us continue with the above setting. In particular, $(X, D) = (\mathcal{X}_0, \mathcal{D}_0)$ is \mathfrak{B} -K-polystable. Then (X, D) admits a conical Kähler-Einstein metric $\omega_X(\mathbf{T})$ with angle $2\pi(1 - (1 - \mathbf{T})/m)$ along the divisor D .

Furthermore, for any sequence $\{\beta_i\} \subset (\epsilon, \mathbf{T})$ satisfying $\beta_i \nearrow \mathbf{T}$, we have

$$\text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\text{Chow}(X, (1 - \beta_i)D), \text{U}(N+1) \cdot \text{Chow}(X, (1 - \mathbf{T})D)) \longrightarrow 0,$$

where $\text{Chow}(X, (1 - \mathbf{T})D)$ is the Chow point corresponding to the cycle obtained via Tian's embedding of $(X, D; \omega_X(\mathbf{T}))$.

Proof. By Theorem 4.1 and the definition of \mathbf{T} , for any $\beta < \mathbf{T}$, the Gromov-Hausdorff limit as $t \rightarrow 0$ of $(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta))$ converges to a weak conical Kähler-Einstein metric on $(X, D; \omega(\beta)) = (\mathcal{X}_0, \mathcal{D}_0; \omega(0, \beta))$. This implies that for each fixed $\beta_i < \mathbf{T}$, there is a $C^\circ \ni t_i \rightarrow 0$ so that

$$(22) \quad \text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}), \text{U}(N+1) \cdot \text{Chow}(X, (1 - \beta_i)D)) < 1/i.$$

It follows from Theorem 4.1 that any subsequence of $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta_i))\}$, there is a Gromov-Hausdorff convergent subsequence. Now suppose there is a subsequence

$$(\mathcal{X}_{t_{i_k}}, \mathcal{D}_{t_{i_k}}; \omega(t_{i_k}, \beta_{i_k})) \xrightarrow{\text{GH}} (Y, E; \omega_Y(\mathbf{T})) \text{ as } k \rightarrow \infty,$$

from which we obtain there are $g_{i_k} \in \text{U}(N+1)$ such that

$$g_{i_k} \cdot \text{Chow}(\mathcal{X}_{t_{i_k}}, (1 - \beta_{i_k})\mathcal{D}_{t_{i_k}}) \longrightarrow \text{Chow}(Y, (1 - \mathbf{T})E),$$

where $\text{Chow}(Y, (1 - \mathbf{T})E)$ is the Chow point corresponding to the Tian's embedding of (Y, E) using the limiting conical Kähler-Einstein metric $\omega_Y(\mathbf{T})$ of angle $2\pi(1 - (1 - \mathbf{T})/m)$ along a \mathbb{Q} -Cartier divisor E . In particular, (Y, E) is \mathbf{T} -K-polystable by [Ber16, Theorem 4.2]. On the other hand, by (22) we have

$$(23) \quad \text{Chow}(Y, (1 - \mathbf{T})E) \in \overline{\text{SL}(N+1) \cdot \text{Chow}(X, D)} \subset \mathbb{P}^{\mathbf{d}, n; N},$$

Suppose $(Y, E) \not\cong (X, D)$, then by [Don12b, Proposition 1] there is a test configuration of (X, D) with central fiber (Y, E) and vanishing generalized Futaki invariant since (Y, E) is \mathbf{T} -K-polystable. This

contradicts our assumption that (X, D) is \mathbf{T} -K-polystable. Hence we must have $(Y, E) \cong (X, D)$. In particular, X admits a weak conical Kähler-Einstein metric with angle $2\pi(1 - (1 - \mathbf{T}))$ along D .

In conclusion, we have

$$(\mathcal{X}_{t_{i_k}}, \mathcal{D}_{t_{i_k}}; \omega(t_{i_k}, \beta_{i_k})) \xrightarrow{\text{GH}} (X, D; \omega_X(\mathbf{T})),$$

which implies

$$\text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\text{Chow}(X, (1 - \beta_i)D), U(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D) \rightarrow 0.$$

Combining with (22), the proof is completed. \square

Remark 6.7. Notice that in the argument above, the existence of the conical Kähler-Einstein metric on \mathcal{X}_{t_i} is needed only for an angle $\beta_i < \mathbf{T}$ instead of \mathbf{T} . So the proof remains *valid* by only assuming that \mathcal{X}_t is \mathbf{T} -K-semistable for any $t \in C^\circ$ instead of being \mathbf{T} -K-polystable.

An immediate consequence is the following.

Corollary 6.8. *$\text{Aut}(X, D)$ is finite. If $\mathbf{T} = 1$, $\text{Aut}(X)$ is reductive.*

Proof. The first part is just Lemma 5.4. The second part follows from [CDS15c, Theorem 6] thanks to the existence of weak Kähler-Einstein metric on X . \square

Let

$$(24) \quad \overline{BO} := \lim_{t \rightarrow 0} \overline{\text{SL}(N + 1) \cdot \text{Chow}(\mathcal{X}_t, \mathcal{D}_t)} \subset \mathbb{P}^{\mathbf{d}, n; N}.$$

denote the limiting orbit and

$$O_{\text{Chow}(X, (1 - \mathbf{T})D)} = \text{SL}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D) \text{ and } \overline{O_{\text{Chow}(X, (1 - \mathbf{T})D)}} \subset \mathbb{P}^{\mathbf{d}, n; N}$$

be the $\text{SL}(N + 1)$ -orbit of $\text{Chow}(X, (1 - \mathbf{T})D)$ and its closure. By Corollary 6.8, this allows us to construct an $\text{SL}(N + 1)$ -invariant Zariski open neighborhood

$$(25) \quad \text{Chow}(X, (1 - \mathbf{T})D) \in U \subset \mathbb{P}^{\mathbf{d}, n; N}$$

satisfying the condition (3) in Lemma 3.1. We want to remark that the open neighborhood U is independent of \mathbf{T} (cf. part (1) of Remark 4.7).

Then we have the following

Lemma 6.9. *Let $\{t_i\} \subset C$ be a sequence of points approaching $0 \in C$ and*

$$\{\beta_i\}, \{\beta_i^*\}, \{\beta'_i\} \subset [\epsilon, 1]$$

be three sequences satisfying $\beta_i^ < \beta_i$ for all i .*

- (1) *Assume $\beta_i \rightarrow \mathbf{T}$, $\beta_i^* \rightarrow \mathbf{T}$ and that there is a sequence $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \mid (\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \text{ being } \beta_i\text{-K-polystable}\}$ with $t_i \rightarrow 0$ such that*

$$(26) \quad \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i^*)\mathcal{D}_{t_i}) \xrightarrow{i \rightarrow \infty} U(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D)$$

and for $g_i \in U(N + 1)$

$$(27) \quad g_i \cdot \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}) \xrightarrow{i \rightarrow \infty} \text{Chow}(Y, (1 - \mathbf{T})E).$$

Then $\text{Chow}(Y, (1 - \mathbf{T})E) = g \cdot \text{Chow}(X, (1 - \mathbf{T})D)$ for some $g \in U(N + 1)$.

- (2) *Assume $\beta'_i \nearrow \mathbf{T}$ and that for any fixed i , there is a $g_i \in U(N + 1)$ such that*

$$(28) \quad \text{Chow}(\mathcal{X}_t, (1 - \beta'_i)\mathcal{D}_t) \xrightarrow{t \rightarrow 0} g_i \cdot \text{Chow}(X, (1 - \beta'_i)D)$$

and

$$(29) \quad \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta'_i)\mathcal{D}_{t_i}) \xrightarrow{i \rightarrow \infty} \text{Chow}(Y, (1 - \mathbf{T})E) \in \overline{BO} \setminus O_{\text{Chow}(X, (1 - \mathbf{T})D)}.$$

If $(X, D) \not\cong (Y, E)$, then there exists a sequence $\{t'_i\}$ satisfying $0 < \text{dist}_C(t'_i, 0) < \text{dist}_C(t_i, 0)$ such that

$$\text{Chow}(Y', (1 - \mathbf{T})E') = \lim_{i \rightarrow \infty} \text{Chow}(\mathcal{X}_{t'_i}, (1 - \beta'_i)\mathcal{D}_{t'_i})$$

$$\in \left(\overline{O_{\text{Chow}(X, (1 - \mathbf{T})D)}} \bigcup (U \cap \overline{BO}) \right) \setminus O_{\text{Chow}(X, (1 - \mathbf{T})D)} \subset \mathbb{P}^{\mathbf{d}, n; N}.$$

where $\text{dist}_C : C \times C \rightarrow \mathbb{R}$ is a fixed continuous distance function on C .

Proof of Lemma 6.9. To prove *part 1*), one first notices that (26) together with Lemma 4.9 imply that (X, D) is \mathbf{T} -K-polystable. We will show that under the above assumption and

$$\text{Chow}(Y, (1 - \mathbf{T})E) \notin \text{U}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D),$$

then one can construct a *new* sequence $\{\beta''_i\}$ satisfying $\beta''_i \in [\beta_i^*, \beta_i]$ such that

$$\begin{aligned} \text{Chow}(Y', (1 - \mathbf{T})E') &= \lim_{i \rightarrow \infty} \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta''_i)\mathcal{D}_{t_i}) \\ &\in \left(\overline{O_{\text{Chow}(X, (1 - \mathbf{T})D)}} \bigcup (U \cap \overline{BO}) \right) \setminus O_{\text{Chow}(X, (1 - \mathbf{T})D)} \subset \mathbb{P}^{\mathbf{d}, n; N}. \end{aligned}$$

On the other hand, Lemma 4.9 implies

$$(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta''_i)) \xrightarrow{\text{GH}} (Y', E'; \omega_{Y'}(\mathbf{T})),$$

thus (Y', E') admits weak Kähler-Einstein metric with angle $2\pi(1 - (1 - \mathbf{T})/m)$ along E' and hence \mathbf{T} -K-polystable. These allow one to construct either a test configuration of (X, D) with central fiber (Y', E') and vanishing generalized Futaki invariant or a test configuration of (Y', E') with central fiber (X, D) and vanishing generalized Futaki invariant, contradicting to the fact that both (X, D) and (Y', E') are \mathbf{T} -K-polystable. So we must have

$$\text{Chow}(Y, (1 - \mathbf{T})E) = g \cdot \text{Chow}(X, (1 - \mathbf{T})D)$$

for some $g \in \text{U}(N + 1)$.

Now we proceed to the construction of $\{\beta''_i\}$. Let

$$B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1) \Subset U$$

be the radius ϵ_1 *open* balls with respect to the distance function (21) and U be given as in (25).

By shrinking the pointed curve $(0 \in C)$ if necessary, we may assume that

$$(30) \quad \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i^*)\mathcal{D}_{t_i}) \in \text{U}(N + 1) \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1)$$

for all i thanks to our assumption (26). On the other hand, by our assumption that $(X, D) \not\cong (Y, E)$, and we may assume (Y, E) is not in the closure of the orbit of (X, D) (otherwise, we can just let $\beta''_i = \beta_i$), then there is an $\epsilon_1 > 0$ such that

$$\text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}), O_{\text{Chow}(X, (1 - \mathbf{T})D)}) > \epsilon_1 \text{ for } i \gg 1.$$

By the β -continuity of $\tau(\cdot, t_i)$ for each fixed $i \gg 1$, for any $0 < \varepsilon < \epsilon_1$ there is XW: correction

$$(31) \quad \beta''_{i,k} = \sup \left\{ \beta \in (\beta_i^*, \beta_i) \mid \tau(\cdot, t_i)|_{(\beta_i^*, \beta)} \subset B(O_{\text{Chow}(X, (1 - \mathbf{T})D)}, \varepsilon/2^k) \cup \text{U}(N + 1) \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1) \right\}$$

where $B(O_{\text{Chow}(X, (1 - \mathbf{T})D)}, \varepsilon/2^k)$ is the $\varepsilon/2^k$ -tubular neighbourhood of $O_{\text{Chow}(X, (1 - \mathbf{T})D)}$, that is, $\beta''_{i,k}$ is the smallest β such that $\tau(\cdot, t_i)$ escapes $B(O_{\text{Chow}(X, (1 - \mathbf{T})D)}, \varepsilon/2^k) \cup \text{U}(N + 1) \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1)$. Clearly, we have $\beta''_{i,k+1} \leq \beta''_{i,k}$. Now if

$$\tau(\beta''_{i,0}, t_i) \in \text{SL}(N + 1) \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1)$$

we let $\beta''_i = \beta''_{i,0}$, otherwise, we let $\beta''_i = \beta''_{i,k}$ where $\beta''_{i,k}$ is the *first* number satisfying

$$\tau(\beta''_{i,k}, t_i) \in \text{SL}(N + 1) \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1).$$

Such k exists because of (30). Now by our construction, there is a $g_i \in \text{SL}(N + 1)$ such that

$$(32) \quad \tau(\beta''_i, t_i) \in g_i \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1).$$

We let

$$M_i = \inf \{ \text{Tr}(g^* g) \mid g \in \text{SL}(N + 1) \text{ such that (32) is satisfied} \} + 1$$

and by passing through a subsequence we may assume $\text{Tr}(g_i^* g_i) \leq M_i$. Then we have the following dichotomy:

Case 1. there is a subsequence $\{M_{i_l}\}$ such that $|M_{i_l}| < M$ for some constant M independent of i . Then we claim that

$$\{\tau(\beta''_{i_l}, t_{i_l}) = \text{Chow}(\mathcal{X}_{t_{i_l}}, (1 - \beta''_{i_l})\mathcal{D}_{t_{i_l}})\}$$

is the subsequence we want, and its limit $\text{Chow}(Y', (1 - \mathbf{T})E')$ lies in

$$(U \cap \overline{BO}) \setminus O_{\text{Chow}(X, (1 - \mathbf{T})D)}.$$

To see this, one only needs to notice that it follows from our construction of β''_{i_l} that

$$\text{dist}_{\mathbb{P}^{\mathbf{d},n;N}}(\tau(\beta''_{i_l}, t_i), O_{\text{Chow}(X, (1-\mathbf{T})D)})$$

is *uniformly* bounded from below by some $\varepsilon/2^k$, since there is a $k = k(M)$ such that

$$\left\{ z \in \mathbb{P}^{\mathbf{d},n;N} \mid \text{dist}_{\mathbb{P}^{\mathbf{d},n;N}}(z, g \cdot \text{Chow}(X, (1-\mathbf{T})D)) \leq \varepsilon/2^{k(M)} \text{ and } |g| < M \right\} \subset \text{SL}(N+1) \cdot U.$$

Case 2. $|M_i| \rightarrow \infty$. If that happens, let us replace ε by $\varepsilon/2$ in (31) and repeat the above process, if for the new sequence $\{M_i^{[1]}\} \subset \mathbb{R}$ there is a bounded subsequence $\{M_{i_l}^{[1]}\}$ then we reduce to the *Case 1*, otherwise, we keep on repeating this process. Then either we stop at a finite stage or this becomes an infinite process. If we stop at a finite stage, then we obtain our subsequence as before, if the process never terminates, we claim that we are able to extract a subsequence whose limit $\text{Chow}(Y', (1-\mathbf{T})E')$ lands in the *boundary*

$$\partial \overline{O_{\text{Chow}(X, (1-\mathbf{T})D)}} = \overline{O_{\text{Chow}(X, (1-\mathbf{T})D)}} \setminus O_{\text{Chow}(X, (1-\mathbf{T})D)}.$$

This is because by choosing a diagonal sequence we will have

$$\text{dist}_{\mathbb{P}^{\mathbf{d},n;N}}(\tau(\beta''_{i_k}^{[k]}, t_{i_k}), O_{\text{Chow}(X, (1-\mathbf{T})D)}) < \varepsilon/2^k \rightarrow 0,$$

so we know

$$z := \lim_{k \rightarrow \infty} \tau(\beta''_{i_k}^{[k]}, t_{i_k}) \in \overline{O_{\text{Chow}(X, (1-\mathbf{T})D)}}.$$

On the other hand, if $z \in O_{\text{Chow}(X, (1-\mathbf{T})D)}$, then

$$z = g \cdot \text{Chow}(X, (1-\mathbf{T})D)$$

for some $g \in \text{SL}(N+1)$. In particular, $g \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1)$ contains a neighborhood of z . However, this violates the assumption that $|M_{i_k}^{[k]}| \rightarrow \infty$ as $k \rightarrow \infty$. Hence our proof is completed.

The proof of *part 2*) is similar. Contrast to the *part 1*), we will vary t instead of β in $\tau(\beta, t)$. First by our assumption (29) together with Lemma 4.9, (Y, E) is \mathbf{T} -K-polystable hence

$$\text{Chow}(Y, (1-\mathbf{T})E) \notin \partial \overline{O_{\text{Chow}(X, (1-\mathbf{T})D)}}.$$

So there is an $\epsilon_1 > 0$ such that

$$\text{dist}_{\mathbb{P}^{\mathbf{d},n;N}}(\text{Chow}(\mathcal{X}_{t_i}, (1-\beta'_i)\mathcal{D}_{t_i}), O_{\text{Chow}(X, (1-\mathbf{T})D)}) > \epsilon_1 \text{ for } i \gg 1.$$

On the other hand, by our assumption (28) and Lemma 6.6 we have for any *fixed* β'_i with $i \gg 1$, there is a $0 < s_i \in \mathbb{R}$ such that

$$\text{Chow}(\mathcal{X}_{t'_i}, (1-\beta'_i)\mathcal{D}_{t'_i}) \in \text{U}(N+1) \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1)$$

for any t satisfying $0 < \text{dist}_C(t, 0) < s_i$, since

$$\text{U}(N+1) \cdot \text{Chow}(X, (1-\beta'_i)D) \xrightarrow{i \rightarrow \infty} \text{U}(N+1) \cdot \text{Chow}(X, (1-\mathbf{T})D)$$

inside $\mathbb{P}^{\mathbf{d},n;N}/\text{U}(N+1)$.

By the t -continuity of $\tau(\beta'_i, \cdot)$ for each fixed $i \gg 1$, for any $\varepsilon < \epsilon_1/2$ there is

$$(33) \quad s_{i,k} := \sup \left\{ s \in [0, |t_i|] \mid \tau(\beta'_i, \cdot)|_{B_C(0,s)} \subset B(O_{\text{Chow}(X, (1-\mathbf{T})D)}, \varepsilon/2^k) \cup \text{U}(N+1) \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1) \right\}$$

where $|t_i| := \text{dist}_C(t_i, 0)$ and $B_C(0, s) := \{t \in C \mid \text{dist}_C(t, 0) \leq s\}$. Then $s_{i,k} = |t_{i,k}|$ is the smallest distance needed for t so that $\tau(\beta'_i, t)$ escapes $B(O_{\text{Chow}(X, (1-\mathbf{T})D)}, \varepsilon/2^k) \cup \text{U}(N+1) \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1)$. Clearly, we have $s_{i,k+1} < s_{i,k}$. Now if

$$\tau(\beta'_i, t_{i,0}) \in \text{SL}(N+1) \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1)$$

we let $t'_i = t_{i,0}$, otherwise, we let $t'_i = t'_{i,k}$ where $t'_{i,k}$ is the *first* point in C satisfying

$$\tau(\beta'_i, t'_{i,k}) \in \text{SL}(N+1) \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1).$$

Such a process must terminate in *finite* steps by (28). Now we define $M_i \in \mathbb{R}$ to be

$$M_i := \inf_{g_i} \{ \text{Tr}(g_i^* g_i) + 1 \mid \tau(\beta'_i, t'_i) \in g_i \cdot B(\text{Chow}(X, (1-\mathbf{T})D), \epsilon_1) \}.$$

Then again we have two situations exactly the same as in the proof of part one depending on $\{M_i\}$ being bounded or not. Replacing β_i'' by t_i' in the argument for Part 1), one see that the rest of the proof is a verbatim, which we will skip. Thus the proof of the Lemma is completed. \square

Remark 6.10. Notice that when $\mathbf{T} = 1$ and both $\beta_i, \beta_i^* \leq 1$, $\forall i$ then Lemma 6.9 and its proof imply a slight variation of the following form.

Let

$$(34) \quad \begin{array}{ccc} \pi_1 : \mathbb{P}^{\mathbf{d}, n; N} = \mathbb{P}^{d, n; N} \times \mathbb{P}^{\delta, n-1; N} & \longrightarrow & \mathbb{P}^{d, n; N} \\ (\text{Chow}(X), \text{Chow}(D)) & \longmapsto & \text{Chow}(X) \end{array}$$

be the projection to the first factor.

- (1) Assume $\beta_i \rightarrow 1, \beta_i^* \rightarrow 1$ and that there is a sequence $\{(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \mid (\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \text{ being } \beta_i\text{-K-polystable}\}$ with $t_i \rightarrow 0$ such that

$$(35) \quad \pi_1(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i^*)\mathcal{D}_{t_i})) \xrightarrow{i \rightarrow \infty} U(N+1) \cdot \text{Chow}(X) \subset \mathbb{P}^{d, n; N}$$

and for $g_i \in U(N+1)$

$$(36) \quad \pi_1(g_i \cdot \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i})) \xrightarrow{i \rightarrow \infty} \text{Chow}(Y) \in \mathbb{P}^{d, n; N}.$$

Then $\text{Chow}(Y) = g \cdot \text{Chow}(X)$ for some $g \in U(N+1)$.

- (2) Assume $\beta_i' \nearrow 1$ and that for any fixed i , there is a $g_i \in U(N+1)$ such that

$$(37) \quad \text{Chow}(\mathcal{X}_t, (1 - \beta_i')\mathcal{D}_t) \xrightarrow{t \rightarrow 0} g_i \cdot \text{Chow}(X, (1 - \beta_i')D) \in \mathbb{P}^{\mathbf{d}, n; N}$$

and

$$(38) \quad \pi_1(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i')\mathcal{D}_{t_i})) \xrightarrow{i \rightarrow \infty} \text{Chow}(Y) \in \overline{BO} \setminus O_{\text{Chow}(X)} \subset \mathbb{P}^{d, n; N}.$$

If $X \not\cong Y$, then there exists a sequence $\{t_i'\}$ satisfying $0 < \text{dist}_C(t_i', 0) < \text{dist}_C(t_i, 0)$ such that

$$\begin{aligned} \text{Chow}(Y') &= \lim_{i \rightarrow \infty} \pi_1(\text{Chow}(\mathcal{X}_{t_i'}, (1 - \beta_i')\mathcal{D}_{t_i'})) \\ &\in \left(\overline{O_{\text{Chow}(X)}} \bigcup (U \cap \overline{BO}) \right) \setminus O_{\text{Chow}(X)} \subset \mathbb{P}^{d, n; N}. \end{aligned}$$

where $\text{dist}_C : C \times C \rightarrow \mathbb{R}$ is a fixed *continuous distance* function on C .

Now we are ready to prove the openness.

Proposition 6.11. *Let $(\mathcal{X}, \mathcal{D}; \mathcal{L}) \rightarrow C$ be Kähler-Einstein degeneration of index (r, \mathfrak{B}) as in Definition 6.1 with $r = r(\mathcal{X}, \mathcal{D})$ being the uniform index as in Theorem 4.1(3). Then $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) \subset [\epsilon, \mathfrak{B}]$ is an open set.*

Proof. Let us assume $\mathbf{T} \in \mathbf{B}_r(\mathcal{X}, \mathcal{D})$, then by fixing a local basis $\{s_i\}$ for $\pi_* \omega_{\mathcal{X}/C}^{-\otimes r}$ we have

$$(39) \quad \text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\text{Chow}(\mathcal{X}_t, (1 - \mathbf{T})\mathcal{D}_t), U(N+1) \cdot \text{Chow}(X, (1 - \mathbf{T})D)) \rightarrow 0 \text{ as } t \rightarrow 0.$$

Now we claim that there is a $\delta > 0$ such that $[\epsilon, \mathbf{T} + \delta) \subset \mathbf{B}_r(\mathcal{X}, \mathcal{D})$. Suppose not, for any k , there is a $\mathbf{T} < \beta_k < \mathbf{T} + 1/k$ and a sequence $\{t_{i,k}\}_{k=1}^\infty$:

$$\text{Chow}(\mathcal{X}_{t_{i,k}}, (1 - \beta_k)\mathcal{D}_{t_{i,k}}) \xrightarrow{i \rightarrow \infty} \text{Chow}(Y_k, (1 - \beta_k)E_k) \notin U \subset \mathbb{P}^{\mathbf{d}, n; N}$$

with $U \subset \mathbb{P}^{\mathbf{d}, n; N}$ being the $\text{SL}(N+1)$ -invariant Zariski open neighborhood of $\text{Chow}(X, (1 - \mathbf{T})D)$ constructed in Lemma 3.1, since (X, D) is also β_k -K-polystable because of $\beta_k \in [\epsilon, \mathfrak{B}]$ and Lemma 2.4. For any fixed i , we can pick up $k_i \gg 0$ such that

$$\text{Chow}(\mathcal{X}_{t_{i,k_i}}, (1 - \beta_{k_i})\mathcal{D}_{t_{i,k_i}}) \notin U(N+1) \cdot B(\text{Chow}(X, (1 - \mathbf{T})D), \epsilon_1).$$

Now let us introduce the *diagonal* sequence

$$\{\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}) := \text{Chow}(\mathcal{X}_{t_{i,k_i}}, (1 - \beta_{k_i})\mathcal{D}_{t_{i,k_i}})\}_{i=0}^\infty.$$

Then by Theorem 4.1, after passing to a subsequence if necessary, we obtain a new sequence, which by abuse of notation will still be denoted by $\beta_i \searrow \mathbf{T}$ and $t_i \rightarrow 0$, such that

$$(40) \quad \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}) \longrightarrow \text{Chow}(Y, (1 - \mathbf{T})E) \notin O_{\text{Chow}(X, (1 - \mathbf{T})D)}.$$

But this violates the first part of Lemma 6.9(1) with $\beta_i^* = \mathbf{T} \forall i$. \square

Next we prove the closedness.

Proposition 6.12. *Let $(\mathcal{X}, \mathcal{D}) \rightarrow C$ be a family satisfying the condition of Proposition 6.11. Suppose further that $\mathcal{X} \rightarrow C$ is a family of \mathfrak{B} -K-polystable varieties. Then $\mathbf{B}(\mathcal{X}, \mathcal{D}) \subset [\epsilon, \mathfrak{B}]$ is also closed with respect to the induced topology, hence $\mathbf{B}(\mathcal{X}, \mathcal{D}) = [\epsilon, \mathfrak{B}]$.*

Proof. By our assumption, for every $t \in C^\circ$, $(\mathcal{X}_t, \mathcal{D}_t)$ is a smooth Fano pair with $\mathcal{D}_t \in |-mK_{\mathcal{X}_t}|$. Since \mathcal{X}_t is \mathfrak{B} -K-polystable, hence it is β -K-polystable for $\beta \in [\epsilon, \mathfrak{B}]$ by Lemma 2.4. As $(\mathcal{X}_t, \mathcal{D}_t)$ are smooth, by Theorem 4.1 and [SW12, Proposition 2.2] [LS14, Proposition 1.7] it admits a unique conical Kähler-Einstein metric ω_t solving

$$\text{Ric}(\omega(t, \beta)) = \beta \omega(t, \beta) + \frac{1 - \beta}{m} [\mathcal{D}_t]$$

with angle $2\pi(1 - (1 - \beta)/m)$ along \mathcal{D}_t for any $\beta \in [\epsilon, \mathfrak{B}]$. By Theorem 4.1 and definition of \mathbf{T} , for any fixed $\beta < \mathbf{T}$, we have

$$(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta)) \xrightarrow{\text{GH}} (\mathcal{X}_0, \mathcal{D}_0; \omega(0, \beta)) \text{ as } t \rightarrow 0.$$

By Lemma 6.6, for any sequence $\beta_i \nearrow \mathbf{T}$ we have

$$\text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\text{Chow}(X, (1 - \beta_i)D), \text{U}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D)) \rightarrow 0.$$

Our goal is to prove that

$$\text{Chow}(\mathcal{X}_t, (1 - \mathbf{T})\mathcal{D}_t) \rightarrow \text{U}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D) \text{ as } t \rightarrow 0.$$

We will argue by contradiction.

Suppose this is not the case, then there is a subsequence $\{t_i\}_{i=1}^\infty \subset C$, $t_i \rightarrow 0$ as $i \rightarrow \infty$ such that

$$\text{Chow}(\mathcal{X}_{t_i}, (1 - \mathbf{T})\mathcal{D}_{t_i}) \rightarrow \text{Chow}(Y, (1 - \mathbf{T})E) \notin \text{U}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D).$$

By the continuity of $\tau(\cdot, t_i)$ at \mathbf{T} for each fixed i (cf. Lemma 6.5), there is a consequence $\{\beta'_i\}_{i=1}^\infty \subset (\epsilon_0, \mathbf{T})$ such that $\beta'_i \nearrow \mathbf{T}$ and

$$(41) \quad \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta'_i)\mathcal{D}_{t_i}) \rightarrow \text{Chow}(Y, (1 - \mathbf{T})E) \notin \text{U}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D) \text{ as } i \rightarrow \infty.$$

We claim that $\text{Chow}(Y, (1 - \mathbf{T})E) \in \overline{BO} \setminus \text{SL}(N + 1) \cdot U$. Otherwise, $\text{Chow}(Y, (1 - \mathbf{T})E) \in U$ then

$$\text{Chow}(X, (1 - \mathbf{T})D) \in \overline{\text{SL}(N + 1) \cdot \text{Chow}(Y, (1 - \mathbf{T})E)}.$$

But this violates the fact that (Y, E) is \mathbf{T} -K-polystable by [Ber16, Theorem 4.2], since we can construct a test configuration of (Y, E) with central fiber (X, D) and vanishing generalized Futaki invariant. Hence our claim is proved.

Now we can apply the second part of Lemma 6.9 to obtain a new sequence $\{t'_i\} \subset C^\circ$ satisfying $t'_i \rightarrow 0 \in C$ and

$$(42) \quad \begin{aligned} \text{Chow}(Y', (1 - \mathbf{T})E') &= \lim_{i \rightarrow \infty} \text{Chow}(\mathcal{X}_{t'_i}, (1 - \beta'_i)\mathcal{D}_{t'_i}) \\ &\in \left(\overline{\text{O}_{\text{Chow}(X, (1 - \mathbf{T})D)}} \bigcup (U \cap \overline{BO}) \right) \setminus \text{O}_{\text{Chow}(X, (1 - \mathbf{T})D)} \subset \mathbb{P}^{\mathbf{d}, n; N}, \end{aligned}$$

which contradicts to the fact that both (Y', E') and (X, D) are \mathbf{T} -K-polystable by the same reason as above. Thus the proof is completed. \square

Remark 6.13. We remark an interesting point of the proof is that in the proof of Proposition 6.11, we have only used the continuity of $\tau(\cdot, t)$ for each fixed t . In particular, its continuity of τ with respect to the variable t is not used. Contrast to this, the continuity of $\tau(\beta, \cdot)$ with respect to t is what we use in the proof of Proposition 6.12.

We note that by this point, we have already established the following.

Corollary 6.14. *Theorem 1.2 holds under an additional assumption that \mathcal{X}_t is β -K-polystable for all $t \in C^\circ$.*

7. K-SEMISTABILITY OF THE NEARBY FIBERS

7.1. Orbit of K-semistable points. In this subsection, we extend our continuity method to study the *uniqueness* of K-polystable Fano varieties that a K-semistable Fano manifold can degenerate to, which will also be needed in the proof of our main theorem.

Let X be a smooth Fano manifold, and $D \in |-mK_X|$ be a smooth divisor for $m \geq 2$. Assume X is \mathbf{T} -K-semistable with respect to D . By Theorem 4.1, we know that for any sequence $\beta_i \nearrow \mathbf{T}$, after possibly passing to a subsequence (, which by abusing of notation will still be denoted by $\beta_i \nearrow \mathbf{T}$), there exists a log \mathbb{Q} -Fano pair (X_0, D_0) which is the Gromov-Hausdorff limit of the conical Kähler-Einstein metric $(X, D; \omega(\beta_i))$, that is,

$$\text{Chow}(X, (1 - \beta_i)D) \longrightarrow \text{U}(N + 1) \cdot \text{Chow}(X_0, (1 - \mathbf{T})D_0) \in \overline{\text{O}_{\text{Chow}(X, (1 - \mathbf{T})D)}} \text{ as } i \rightarrow \infty$$

with X_0 being \mathbf{T} -K-polystable, where

$$\overline{\text{O}_{\text{Chow}(X, (1 - \mathbf{T})D)}} = \text{the closure of } \text{SL}(N + 1) \cdot \text{Chow}(X, (1 - \mathbf{T})D) \subset \mathbb{P}^{\mathbf{d}, n; N}.$$

In particular, (X_0, D_0) admits a weak conical Kähler-Einstein metric $\omega(\mathbf{T})$ with cone angle $2\pi(1 - (1 - \mathbf{T})/m)$ along the divisor $D_0 \subset X_0$.

Lemma 7.1. *The limit is independent of the choice of the sequence $\{\beta_i\}$ in the sense that for every sequence $\beta_i \nearrow \mathbf{T}$,*

$$(X, D; \omega(\beta_i)) \xrightarrow{\text{GH}} (X_0, D_0; \omega(\mathbf{T})).$$

Proof. The existence of a weak conical Kähler-Einstein metric $\omega(\mathbf{T})$ on (X_0, D_0) allows us to construct a test configuration $(\mathcal{X}, \mathcal{D}; \mathcal{L})$ of (X, D) with central fiber (X_0, D_0) since $\text{Aut}(X_0, D_0)$ is reductive by Theorem 4.1. Now our claim follows by applying Lemma 6.9 (1) to the family $(\mathcal{X}, \mathcal{D}; \mathcal{L})$. \square

Theorem 7.2. *Suppose X is a smooth K-semistable Fano manifold and $D_0 \in |-m_0K_X|$ and $D_1 \in |-m_1K_X|$ are two smooth divisors. Let X_0 and X_1 be the limits defined as in Lemma 7.1 with $\mathbf{T} = 1$, then $X_0 \cong X_1$.*

Proof. By introducing a third divisor in $|-mK_X|$ with $m = \text{lcm}(m_0, m_1)$, we may assume $rm_0 = m_1$ for a positive integer r . By Bertini's Theorem, we may choose $\{D_t\}_{t \in [0, 1]} \subset |-mK_X|$ to be a continuous path joining rD_0 and D_1 such that

- the path $\{D_t\}$ lies in an algebraic arc $C \subset |-mK_X|$ with corresponding family $\mathcal{D} \rightarrow C$;
- D_t is smooth for all $t \neq 0$.

By assumption, X is K-semistable, hence (X, D_t) are β -K-stable for all $(\beta, t) \in (0, 1) \times (0, 1]$. In particular, $\{(X, D_t)\}$ admit conical Kähler-Einstein metric $\omega(t, \beta)$, $\forall (\beta, t) \in (0, 1) \times [0, 1]$ by Corollary 4.2, using Tian's embedding we can similarly define a map

$$(43) \quad \begin{array}{ccc} \sigma : (0, 1) \times (0, 1] & \longrightarrow & \mathbb{P}^{\mathbf{d}, n; N} \\ (\beta, t) & \longmapsto & \text{Chow}(X, (1 - \beta)D_t) \end{array}$$

using a prefixed basis of $H^0(X, \mathcal{O}_X(-rK_X))$. By Proposition 4.8 and [Don12a, Theorem 2], σ is continuous on $(0, 1) \times (0, 1]$. We claim that $q \circ \sigma$ is continuous on $(0, 1) \times [0, 1]$ with $q : \mathbb{P}^{\mathbf{d}, n; N} \rightarrow \mathbb{P}^{\mathbf{d}, n; N}/\text{U}(N + 1)$. For fixed $\beta \in (0, 1)$, we can deduce the continuity of $\sigma(\beta, \cdot)$ at 0 by applying Corollary 6.14 to the product family $(\mathcal{X} = X \times C, \mathcal{D}) \rightarrow C$ with $(\mathcal{X}_t, \mathcal{D}_t) = (X, D_t)$.

Thus all we need to show is

$$(44) \quad \lim_{\beta \rightarrow 1} \text{dist}_{\mathbb{P}^{\mathbf{d}, n; N}}(\tilde{\sigma}(\beta, t), \text{U}(N + 1) \cdot \text{Chow}(X_0)) = 0, \quad \forall t \in [0, 1]$$

where $\tilde{\sigma} := \pi_1 \circ \sigma$ with π_1 being given in (34). To achieve that, let $\tilde{q} : \mathbb{P}^{\mathbf{d}, n; N} \rightarrow \mathbb{P}^{\mathbf{d}, n; N}/\text{U}(N + 1)$ then Lemma 7.1 allows us to introduce

$$\lim_{\beta \rightarrow 1} \tilde{q} \circ \tilde{\sigma}(\beta, t) = \text{U}(N + 1) \cdot \text{Chow}(X_t) \in \mathbb{P}^{\mathbf{d}, n; N}/\text{U}(N + 1), \text{ for } t \in [0, 1]$$

with X_t being a \mathbb{Q} -Fano variety admitting weakly Kähler-Einstein metric for each $t \in [0, 1]$. Let $\mathcal{X}_1 \rightarrow \mathbb{A}^1$ be a test configuration with central fiber X_1 and $\text{Chow}(X_1) \in U \subset \mathbb{P}^{\mathbf{d}, n; N}$ be the open neighborhood constructed for the family $\mathcal{X}_1 \rightarrow \mathbb{A}^1$ via Lemma 3.1.

Now suppose (44) does not hold, i.e. there is a $t_0 \in [0, 1]$ such that

$$\lim_{\beta \rightarrow 1} \tilde{\sigma}(\beta, t_0) = \text{Chow}(X_{t_0}) \notin U \cdot \text{Chow}(X_1) .$$

Then by applying the continuity of $\tilde{q} \circ \tilde{\sigma}(\beta, \cdot)$ with respect to $t \in [0, 1]$ for fixed β the same way as in the proof of Lemma 6.9(2), we can construct a new sequence $\{(\beta_i, t_i)\}_{i=1}^\infty \subset (0, 1] \times [t_0, 1]$ such that $\beta_i \nearrow 1$ as $i \rightarrow \infty$ and

$$\text{Chow}(Y) = \lim_{i \rightarrow \infty} \tilde{\sigma}(\beta_i, t_i) \in \left(\overline{O_{\text{Chow}(X_1)}} \bigcup (U \cap \partial \overline{O_{\text{Chow}(X)}}) \right) \setminus O_{\text{Chow}(X_1)} \subset \mathbb{P}^{d, n; N} ,$$

with both X_1 and $Y (\not\cong X_1)$ being K-polystable, which is impossible. Hence our proof is completed. \square

7.2. Zariski Openness of K-semistable varieties. In this section, we will study the Zariski openness of the locus of the smoothable K-semistable varieties inside Chow schemes. This needs a combination of the continuity method with the algebraic result in Appendix 9.1.

Let

$$\begin{array}{ccc} (\mathcal{X}, \mathcal{D}) & \xrightarrow{\iota} & \mathbb{P}^N \times \mathbb{P}^N \times S \\ \downarrow \pi & & \downarrow \\ S & \xlongequal{\quad} & S \end{array}$$

be a flat family of \mathbb{Q} -Fano varieties over a smooth base S (not necessarily complete) and $\mathcal{D} \in |-mK_{\mathcal{X}}|$ be an irreducible divisor defined by a section $s_{\mathcal{D}} \in \Gamma(S, \mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}}))$. Let us assume further that $\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}})$ is relatively very ample and ι is the embedding induced by a prefixed basis $\{s_i(t)\}_{i=0}^N \subset \Gamma(S, \pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/S}))$, in particular $\iota^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/S})$. Then we have the following

Theorem 7.3. *Let $(\mathcal{X}, \mathcal{D}) \rightarrow C$ be the family over a smooth curve such that $(\mathcal{X}_t, \mathcal{D}_t)$ is smooth for $t \in C^\circ$ and $(\mathcal{X}_t, \frac{1}{m}\mathcal{D}_t)$ is a klt for all $t \in C$. Assume $(\mathcal{X}_0, \mathcal{D}_0)$ is \mathfrak{B} -K-semistable. Then there is a Zariski open neighborhood $0 \in C^* \subset C$ such that $(\mathcal{X}_t, \mathcal{D}_t)$ is \mathfrak{B} -K-semistable for $t \in C^*$. Furthermore, if $(\mathcal{X}_0, \mathcal{D}_0)$ is \mathfrak{B} -K-polystable and has only finitely many automorphisms, then $(\mathcal{X}_t, \mathcal{D}_t)$ is \mathfrak{B} -K-polystable after a possibly further shrinking of C^* .*

Definition 7.4. For every $t \in S$, we define the *K-semistable threshold* as follows

$$\text{kst}(\mathcal{X}_t, \mathcal{D}_t) := \sup \{ \beta \in [0, \mathfrak{B}] \mid (\mathcal{X}_t, \mathcal{D}_t) \text{ is } \beta\text{-K-semistable} \} .$$

By Theorem 4.1, testing β -K-semistability for \mathcal{X}_t , $\forall t \in S$ is reduced to test for all 1-PS inside $\text{SL}(N+1)$ for a fixed sufficiently large \mathbb{P}^N . This implies that $\text{kst}(\mathcal{X}_t, \mathcal{D}_t)$ is a constructible function of t (cf. Proposition 7.5 below). By Remark 6.4, we know $(\mathcal{X}_t, \mathcal{D}_t)$ is β -K-stable for all $\beta \in (0, \beta_0]$. This together with Lemma 2.4 in particular imply that $\text{kst}(\mathcal{X}_t, \mathcal{D}_t)$ is actually a *maximum* for every $t \in S$.

Then we have the following Proposition whose proof will be given in the Section 9.1, Proposition 9.4.

Proposition 7.5. *$\text{kst}(\mathcal{X}_t, \mathcal{D}_t)$ defines a constructible function on S , i.e. $S = \sqcup_i S_i$ is a union of finite constructible sets $\{S_i\}$, on which $\text{kst}(\mathcal{X}_t, \mathcal{D}_t)$ are constant.*

Proof of Theorem 7.3. By Proposition 7.5, $\text{kst}(\mathcal{X}_t, \mathcal{D}_t)$ is constant when restricted to each strata S_i . So all we need is that if $t_i \rightarrow 0$ and $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ strictly \mathbf{T} -K-semistable then

$$\mathbf{T} = \text{kst}(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \geq \text{kst}(X, D) = \mathfrak{B} .$$

Suppose this is not the case, we have $\mathfrak{B} > \mathbf{T}$ and we seek for a contradiction. First, we claim for any sequence $t_i \rightarrow 0$, after passing to a subsequence which by abusing of notation still denoted by $\{t_i\}$, we can find a sequence $\{\beta_i^*\} \nearrow \mathbf{T}$ such that

$$(45) \quad \text{dist}_{\mathbb{P}^{d, n; N}}(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i^*)\mathcal{D}_{t_i}), U(N+1) \cdot \text{Chow}(X, (1 - \mathbf{T})D)) \rightarrow 0 .$$

In fact, since we have already established Theorem 1.2 under the extra assumption that the nearby points are all β -K-polystable (see Corollary 6.14), for any fixed $\beta < \mathbf{T}$ we have

$$\text{dist}_{\mathbb{P}^{d, n; N}}(\text{Chow}(\mathcal{X}_t, (1 - \beta)\mathcal{D}_t), U(N+1) \cdot \text{Chow}(X, (1 - \beta)D)) \rightarrow 0 \text{ as } t \rightarrow 0 ,$$

thus Lemma 6.6 implies that

$$\text{dist}_{\mathbb{P}^d, n; N}(\text{Chow}(X, (1 - \beta'_i)D), U(N+1) \cdot \text{Chow}(X, (1 - \mathbf{T})D)) \longrightarrow 0$$

for any sequence $\beta'_i \nearrow \mathbf{T} < \mathfrak{B}$. Since $t_i \rightarrow 0$, for any fixed β'_i there is a $k_i \geq i$ such that

$$\text{dist}_{\mathbb{P}^d, n; N}(\text{Chow}(\mathcal{X}_{t_{k_i}}, (1 - \beta_i)\mathcal{D}_{t_{k_i}}), U(N+1) \cdot \text{Chow}(X, (1 - \beta_i)D)) < 1/i.$$

Now we pick the subsequence $\{t_{k_i}\}$ and define $\beta_{k_i}^* := \beta'_i$, then the sequence $\{\beta_{k_i}^*\}_{i \rightarrow \infty} \nearrow \mathbf{T}$ is a sequence satisfying (45), hence our claim is justified.

On the other hand, for each fixed t_i , let $\beta \nearrow \mathbf{T}$. By Theorem 4.1, we have

$$(46) \quad \text{dist}_{\mathbb{P}^d, n; N}(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta)\mathcal{D}_{t_i}), U(N+1) \cdot \text{Chow}(\tilde{\mathcal{X}}_{t_i}, (1 - \mathbf{T})\tilde{\mathcal{D}}_{t_i})) \longrightarrow 0$$

with $\text{Chow}(\tilde{\mathcal{X}}_{t_i}, (1 - \mathbf{T})\tilde{\mathcal{D}}_{t_i}) \in \overline{\partial \mathcal{O}_{\text{Chow}(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})}}$ and $(\tilde{\mathcal{X}}_{t_i}, (1 - \mathbf{T})\tilde{\mathcal{D}}_{t_i})$ being a \mathbf{T} -K-polystable variety. Now we claim that

$$(47) \quad \text{Chow}(\tilde{\mathcal{X}}_{t_i}, (1 - \mathbf{T})\tilde{\mathcal{D}}_{t_i}) \longrightarrow g \cdot \text{Chow}(X, (1 - \mathbf{T})D) \text{ for some } g \in U(N+1).$$

To see this, one notices that by Theorem 4.1 and Lemma 4.9 after passing to a subsequence there is a sequence $\beta_i \nearrow \mathbf{T}$ such that

$$\text{Chow}(\tilde{\mathcal{X}}_{t_i}, (1 - \beta_i)\tilde{\mathcal{D}}_{t_i}) \longrightarrow \text{Chow}(Y, (1 - \mathbf{T})E),$$

such that (Y, E) is \mathbf{T} -K-polystable. Moreover, we may assume $\beta_i^* < \beta_i, \forall i$ after rearranging. Combining (46) and Lemma 4.9 we know

$$(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta_i)) \xrightarrow{\text{GH}} (Y, E; \omega_Y(\mathbf{T})),$$

where (Y, E) is a log \mathbb{Q} -Fano pair admitting a weak conical Kähler-Einstein metric $\omega_Y(\mathbf{T})$ with angle $2\pi(1 - (1 - \mathbf{T})/m)$ along E . In particular, (Y, E) is \mathbf{T} -K-polystable. By Lemma 6.9(1), we conclude that

$$\text{Chow}(Y, (1 - \mathbf{T})E) = g \cdot \text{Chow}(X, (1 - \mathbf{T})D) \text{ for some } g \in U(N+1).$$

Hence our claim is proved.

To conclude the proof, we notice that the stabilizer group of $\text{Chow}(\mathcal{X}'_i, (1 - \mathbf{T})\mathcal{D}'_i)$ is of positive dimension for each i . Let $\mathfrak{g} = \mathfrak{sl}(N+1)$ be the Lie algebra. By the upper semicontinuity of the dimension of the stabilizer $\mathfrak{g}_{\text{Chow}(\mathcal{X}'_i, (1 - \mathbf{T})\mathcal{D}'_i)}$, we must have $\dim \mathfrak{g}_{\text{Chow}(X, (1 - \mathbf{T})D)} > 0$ contradicting to the fact that the automorphism group of (X, D) is finite for $\mathbf{T} < \mathfrak{B} \leq 1$ (see Corollary 6.8). To prove the last part of the statement, we just notice that under our assumption $(\mathcal{X}'_i, \mathcal{D}'_i)$ has to have finite automorphism groups, which implies

$$(\mathcal{X}'_i, \mathcal{D}'_i) \cong (\mathcal{X}_i, \mathcal{D}_i).$$

Hence our proof is completed for this case. \square

7.3. Proof of Theorem 1.1 and 1.2. Before we start the proof, let us fix a divisor $\mathcal{D} \sim_C -mK_{\mathcal{X}}$ in *general position* for the flat family $\mathcal{X} \rightarrow C$ satisfying the assumption of Theorem 5.2 and $(\mathcal{X}_t, \mathcal{D}_t)$ being smooth for all $t \in C^\circ$.

Proof of Theorem 1.1. First, we notice that (i) is proved in Section 7.2.

To prove (ii), one notices that Theorem 4.1 implies that there exists an r , such that the Gromov-Hausdorff limit of the family $(\mathcal{X}_t, \mathcal{D}_t; \omega(t, \beta_t))$ for any $t \in C$ and $\beta < 1$ can all be embedded into \mathbb{P}^N for $N = N(r, d)$. By putting Proposition 6.11 and 6.12 together, we obtain that for every $\mathfrak{B} < 1$,

$$\mathbf{B}_r(\mathcal{X}, \mathcal{D}) = [\epsilon, \mathfrak{B}]$$

for $(\mathcal{X}, \mathcal{D})$ (See Corollary 6.14). Therefore, their union will contain $[\epsilon, 1]$. In particular, it follows from Lemma 6.6 and Remark 6.7 for $\mathfrak{B} = 1$ that $X = \mathcal{X}_0$ admits a Kähler-Einstein metric. This in particular verifies the first part of (iii).

Now we finish the proof of part (ii). By part (i), after a possible shrinking of C , we may assume that \mathcal{X}_t is K-semistable for every $t \in C^\circ$. For any $t \neq 0$, there is a *unique* K-polystable \mathbb{Q} -Fano $\tilde{\mathcal{X}}_t$ such that $\text{Chow}(\tilde{\mathcal{X}}_t) \in \overline{\mathcal{O}_{\text{Chow}(\mathcal{X}_t)}}$ by Theorem 7.2, which is the Gromov-Hausdorff limit of $(\mathcal{X}_t, \mathcal{D}_t; \omega(\beta))$ as $\beta \rightarrow 1$ and hence admits a weak Kähler-Einstein metric $\tilde{\omega}(t)$ by Theorem 1.2.

We claim that

$$(48) \quad \text{dist}_{\mathbb{P}^d, n; N}(\text{U}(N+1) \cdot \text{Chow}(\tilde{\mathcal{X}}_t), \text{U}(N+1) \cdot \text{Chow}(X)) \longrightarrow 0, \text{ as } i \rightarrow \infty,$$

and hence part (ii) follows. To prove that, let $t_i \rightarrow 0$ be *any* sequence. It follows from the *compactness* of Chow variety of \mathbb{P}^N that after passing to a subsequence if necessary we may assume

$$\text{Chow}(\tilde{\mathcal{X}}_{t_i}) \longrightarrow \text{Chow}(Y) \text{ as } t_i \rightarrow 0.$$

Since

$$(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta)) \xrightarrow{\text{GH}} (\tilde{\mathcal{X}}_{t_i}; \tilde{\omega}(t_i)) \text{ as } \beta \nearrow 1,$$

by Theorem 7.2, there is a sequence $\beta_i \nearrow 1$ such that

$$\text{dist}_{\mathbb{P}^d, n; N}(\pi_1 \circ \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}), \text{U}(N+1) \cdot \text{Chow}(\tilde{\mathcal{X}}_{t_i})) < 1/i,$$

where π_1 is given in (34). In particular, by passing to another subsequence if necessary, we may assume

$$(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}; \omega(t_i, \beta_i)) \xrightarrow{\text{GH}} (Y, \omega_Y)$$

by Lemma 4.9, where Y is a \mathbb{Q} -Fano variety admitting a weak Kähler-Einstein metric ω_Y . This implies that

$$(49) \quad \text{dist}_{\mathbb{P}^d, n; N}(\pi_1 \circ \text{Chow}(\mathcal{X}_{t_i}, (1 - \beta_i)\mathcal{D}_{t_i}), \text{U}(N+1) \cdot \text{Chow}(Y)) \longrightarrow 0, \text{ as } i \rightarrow \infty.$$

On the other hand, by Lemma 2.4 we know $(\mathcal{X}_{t_i}, \mathcal{D}_{t_i})$ is β -K-polystable for any $\beta < 1$. This together with Corollary 6.14 imply that for every fixed $\beta < 1$

$$\text{dist}_{\mathbb{P}^d, n; N}(\text{Chow}(\mathcal{X}_{t_i}, (1 - \beta)\mathcal{D}_{t_i}), \text{U}(N+1) \cdot \text{Chow}(X, (1 - \beta)D)) \longrightarrow 0 \text{ as } i \rightarrow \infty.$$

Therefore, for any fixed β_i there is a $k_i > i$ such that

$$\text{dist}_{\mathbb{P}^d, n; N}(\text{Chow}(\mathcal{X}_{t_{k_i}}, (1 - \beta_i)\mathcal{D}_{t_{k_i}}), \text{U}(N+1) \cdot \text{Chow}(X, (1 - \beta_i)D)) < 1/i.$$

On the other hand, Lemma 6.6 implies that

$$\text{dist}_{\mathbb{P}^d, n; N}(\pi_1 \circ \text{Chow}(X, (1 - \beta)D), \text{U}(N+1) \cdot \text{Chow}(X)) \longrightarrow 0 \text{ as } \beta \rightarrow 1.$$

These imply that if we define $\beta_{k_i}^* := \beta_i < \beta_{k_i}$ then $\beta_{k_i}^* \rightarrow 1$ and

$$(50) \quad \text{dist}_{\mathbb{P}^d, n; N}(\pi_1 \circ \text{Chow}(\mathcal{X}_{t_{k_i}}, (1 - \beta_{k_i}^*)\mathcal{D}_{t_{k_i}}), \text{U}(N+1) \cdot \text{Chow}(X)) \longrightarrow 0 \text{ as } i \rightarrow \infty.$$

By putting together (50) and (49), and applying Remark 6.10 (1), we conclude that $\text{Chow}(Y) \in \text{U}(N+1) \cdot \text{Chow}(X)$, and (48) is established. Thus the proof of part (ii) is completed.

Finally, to finish the proof of part (iii), we can assume \mathcal{X}_t is K-polystable for all $t \in C$ by Theorem 7.3, then by taking $\mathfrak{B} = 1$ we can conclude that $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) = [\epsilon, 1]$. In particular, $(\mathcal{X}_{t_i}; \omega(t_i)) \xrightarrow{\text{GH}} (\mathcal{X}_0; \omega_{\mathcal{X}_0})$. Hence our proof is completed. \square

Proof of Theorem 1.2. Choose a sequence $\beta \nearrow \mathfrak{B}$. Applying Proposition 6.11 and 6.12, we obtain that $\mathbf{B}_r(\mathcal{X}, \mathcal{D}) = [\epsilon, \mathfrak{B}]$. Then by repeating the argument completely parallel to the one given above, we obtain the conclusion. \square

Remark 7.6. We call a \mathbb{Q} -Fano variety to be *smoothable* if there is a projective flat family \mathcal{X} over a smooth curve C such that $K_{\mathcal{X}}$ is \mathbb{Q} -Cartier, anti-ample over C , a general fiber \mathcal{X}_t is smooth and $X \cong \mathcal{X}_0$ for some $0 \in C$. We note that by a standard argument, we can generalize Theorem 1.1, 7.2 and 7.3 to the case that the base is of higher dimension. As a consequence, we can just assume in these theorems that the general fibers are smoothable instead of smooth. These extensions will be frequently used in Section 8.

8. LOCAL GEOMETRY NEAR A SMOOTHABLE K-POLYSTABLE \mathbb{Q} -FANO VARIETY

In this section, we will study the geometric consequences of Theorem 1.1, especially on how to use it to construct a proper moduli space for smoothable K-polystable Fano varieties.

Our setup works for both Chow and Hilbert scheme, but we choose to work with Chow scheme in order to be consistent with the previous sections.

Definition 8.1. We define

$$(51) \quad Z := \left\{ \text{Chow}(Y) \left| \begin{array}{l} Y \subset \mathbb{P}^N \text{ be a smooth Fano manifold with} \\ \deg Y = d \text{ and } \mathcal{O}_{\mathbb{P}^N}(1)|_Y \cong K_Y^{-\otimes r}. \end{array} \right. \right\} \subset \mathbb{P}^{d,n;N}.$$

By the boundedness of smooth Fano manifolds with fixed dimension (see [KMM92]), we may choose $N \gg 1$ such that Z includes all such Fano manifolds. Now let $\bar{Z} \subset \mathbb{P}^{d,n;N}$ be the closure of $Z \subset \mathbb{P}^{d,n;N}$ and Z° be the open set of \bar{Z} that parametrizes the K-semistable \mathbb{Q} -Fano subvariety Y such that $\omega_Y^{[-r]} \sim \mathcal{O}_{\mathbb{P}^N}(1)|_Y$. Let Z^* be the semi-normalization of Z_{red}° which is the reduction of Z° .

Remark 8.2. By Theorem 4.1, the Gromov-Hausdorff limit of Fano Kähler-Einstein manifolds is automatically in Z° and hence so are the smoothable K-polystable \mathbb{Q} -Fano varieties.

Then we have a commutative diagram

$$(52) \quad \begin{array}{ccccc} \mathcal{X}^* & \xrightarrow{i} & \mathbb{P}^N \times Z^* & \longrightarrow & \mathbb{P}^N \times Z_{\text{red}}^\circ \\ \pi \downarrow & & \downarrow & & \downarrow \\ Z^* & \longrightarrow & Z^* & \longrightarrow & Z_{\text{red}}^\circ \end{array}$$

where \mathcal{X}^* is the universal family over Z^* (see [Kol96, Section I.3]).

Before we state the main result of this section, let us first deduce the following boundedness result which is a consequence of our Theorem 1.1.

Lemma 8.3. *The smoothable K-semistable \mathbb{Q} -Fano varieties with a fixed dimension form a bounded family.*

Proof. We first prove for the statement for K-polystable \mathbb{Q} -Fano varieties. Let X be an n -dimensional smoothable K-polystable \mathbb{Q} -Fano variety and $\mathcal{X} \rightarrow C$ be a smoothing of X with $\mathcal{X}_0 = X$. It follows from Theorem 1.1 that nearby fibers \mathcal{X}_t are all K-semistable, and we can take a $\mathcal{D} \sim_C -mK_{\mathcal{X}/C}$, such that \mathcal{X}_0 is the Gromov-Hausdorff limit of $(\mathcal{X}_{t_i}, (1-\beta_i)\mathcal{D}_{t_i})$ for any sequences $t_i \rightarrow 0$ and $\beta_i \rightarrow 1$.

On the other hand, by the boundedness of smooth Fano varieties, we know that there exists m_0 depending only on n , and a divisor

$$\mathcal{D}^* \sim_{C^\circ} -m_0 K_{\mathcal{X}^\circ/C^\circ},$$

such that \mathcal{D}_t^* is smooth for any $t \in C^\circ$. Since all \mathcal{X}_t are K-semistable, they admit conical Kähler-Einstein metrics $\omega(t, \beta_i)$ with cone angle $2\pi(1 - (1-\beta_i)/m)$ along \mathcal{D}_t^* . By applying Theorem 1.2(iii) for $(\mathcal{X}_t, (1-\beta_i)\mathcal{D}_t^*)$, we know that the Gromov-Hausdorff limit for this family as $t \rightarrow 0$ is also \mathcal{X}_0 . Thus it is a subvariety of a fixed \mathbb{P}^N for some $N \gg 0$ by Theorem 4.1.

In general, if X is smoothable K-semistable \mathbb{Q} -Fano variety, then we know that the closure of its orbit contains a K-polystable \mathbb{Q} -Fano variety X_0 . And as a consequence of volume convergence for Gromov-Hausdorff limit, we obtain that

$$(-K_{X_0})^n = (-K_X)^n$$

are bounded from above; on the other hand the Cartier index of K_X divides the Cartier index of K_{X_0} , which is also bounded from above. Therefore X is contained in a bounded family (see e.g. [HMX14, Corollary 1.8]). \square

Fix a K-polystable \mathbb{Q} -Fano variety X parametrized by a point in Z^* , then it admits a weak Kähler-Einstein metric by Theorem 1.1 from which we deduce that $\text{Aut}(X) \subset \text{SL}(N+1)$ is reductive. Let $\text{Chow}(X)$ be the Chow point for the Tian's embedding of $X \subset \mathbb{P}^N$ after we fix a basis of $H^0(\mathcal{O}_X(-rK_X))$. Then by [Don12b, Proposition 1] or the proof of Lemma 3.1, there is an $\text{Aut}(X)$ -invariant linear subspace $z_0 := \text{Chow}(X) \in \mathbb{P}W \subset \mathbb{P}^{d,n;N}$ so that

$$(53) \quad \mathbb{P}^{d,n;N} = \mathbb{P}(W \oplus \mathbb{C}z_0 \oplus \mathfrak{aut}(X)^\perp) \text{ with } \mathfrak{aut}(X)^\perp \oplus \mathfrak{aut}(X) = \mathfrak{sl}(N+1),$$

where $W \oplus \mathbb{C} \cdot z_0 \oplus \mathfrak{aut}(X)^\perp = (\text{Sym}^d \mathbb{C}^{N+1})^{\otimes(n+1)}$ is a decomposition as $\text{Aut}(X)$ -invariant subspaces.

In particular, this induces a representation $\rho : \text{Aut}(X) \rightarrow \text{SL}(W)$. On the other hand, $\text{Chow}(X)$ is fixed by $\text{Aut}(X)$. We let $\rho_X : \text{Aut}(X) \rightarrow \mathbb{G}_m$ denote the character corresponding to the linearization of $\text{Aut}(X)$ on $\mathcal{O}_{\mathbb{P}^d, n; N}(1)|_{\text{Chow}(X)}$ induced from the embedding $\text{Aut}(X) \subset \text{SL}(N+1)$. Then we can introduce the following

Definition 8.4. A point $z \in \mathbb{P}W$ is *GIT-polystable* (resp. *GIT-semistable*) if z is *polystable* (resp. *semistable*) with respect the linearization $\rho \otimes \rho_X^{-1}$ on $\mathcal{O}_{\mathbb{P}W}(1) \rightarrow \mathbb{P}W$ in the GIT sense.

Our main result of this section is the following:

Theorem 8.5. *There is an $\text{Aut}(X)$ -invariant linear subspace $\mathbb{P}W \subset \mathbb{P}^{d, n; N}$ and a Zariski open neighborhood $\text{Chow}(X) \in U_W \subset \mathbb{P}W \times_{\mathbb{P}^{d, n; N}} Z^*$ such that for any $\text{Chow}(Y) \in U_W$, Y is K -polystable if and only if $\text{Chow}(Y)$ is GIT-polystable with respect to $\text{Aut}(X)$ -action on $\mathbb{P}W \times_{\mathbb{P}^{d, n; N}} Z^*$.*

Moreover, for all GIT-polystable $\text{Chow}(Y) \in U_W$, we have $\text{Aut}(Y) < \text{Aut}(X)$, i.e. the local GIT presentation $U_W // \text{Aut}(X)$ is stabilizer preserving in the sense of [AFS16, Proposition 3.1].

Remark 8.6. As we will see in Corollary 8.12 that we are able to establish the stabilizer preserving property for all *GIT-semistable* $\text{Chow}(Y) \in U_W$. This property is *stronger* than the condition of being *strongly étale* introduced in [AFS16, Definition 2.5].

Let

$$(54) \quad \begin{array}{ccc} \Delta : & Z^* & \longrightarrow \mathbb{P}^{d, n; N} \times Z^* \\ & z & \longmapsto (z, z) . \end{array}$$

be the diagonal morphism, we define $O_{Z^*} := \text{SL}(N+1) \cdot \Delta(Z^*) \subset \mathbb{P}^{d, n; N} \times Z^*$ where $\text{SL}(N+1)$ acts *trivially* on Z^* and acts on $\mathbb{P}^{d, n; N}$ via the action induced from \mathbb{P}^N . This allows us to construct the family of limiting orbits space associated to the family (52) as following:

$$(55) \quad \begin{array}{ccc} \overline{BO}_z \subset \overline{O}_{Z^*} & \xrightarrow{i} & \mathbb{P}^{d, n; N} \times Z^* \\ \downarrow & & \downarrow \pi_{Z^*} \\ z & \in Z^* & \xlongequal{\quad} Z^* \end{array}$$

with $\overline{O}_{Z^*} \subset Z^* \times \mathbb{P}^{d, n; N}$ be the closure and \overline{BO}_z is the union of limiting *broken orbits*. Then by Theorem 1.1 we know that there is a *unique* K -polystable orbit inside \overline{BO}_z . To see this, one only needs to notice that for any $z \in Z^*$, we can always find a smooth curve $f : C \rightarrow Z^*$ that passes through z and the image $f(C)$ meets the dense open locus inside of Z^* corresponding to *smooth K -polystable Fano manifolds* with the *maximal* dimension of its $\text{SL}(N+1)$ -orbit space. Then our claim follows by applying Theorem 1.1 to the pull back family over C .

For a K -polystable point $\text{Chow}(X) \in Z^*$ (corresponding to the Tian's embedding of $X \subset \mathbb{P}^N$ with respect to the Kähler-Einstein metric), by Lemma 3.1, we can find a Zariski neighborhood $\text{Chow}(X) \in U \subset Z^*$ and after a possible shrinking we may assume

$$(56) \quad U \cap \overline{BO}_{\text{Chow}(X)} \text{ contains a unique minimal (cf. Lemma 3.1) orbit } \text{SL}(N+1) \cdot \text{Chow}(X) .$$

By Theorem 7.2 (and its extension in Remark 7.6), every $z \in U$ can be specialized to a K -polystable point \hat{z} unique up to $\text{SL}(N+1)$ -translation. Moreover, we have the following

Lemma 8.7. *Let $\text{Chow}(X) \in U \subset Z^*$ be as above, then there is an analytic open neighborhood $\text{Chow}(X) \in U^{\text{ks}}$ such that for any K -semistable points $z \in U^{\text{ks}}$, we can specialize it to a K -polystable point $\hat{z} \in U$ via a 1-PS $\lambda \subset \text{SL}(N+1)$. Moreover, if $\lim_{i \rightarrow \infty} z_i = \text{Chow}(X)$, then*

$$\lim_{i \rightarrow \infty} \text{dist}_{\mathbb{P}^{d, n; N}}(\text{Chow}(\mathcal{X}_{z_i}, \omega_{\text{KE}}(\hat{z}_i)), U(N+1) \cdot \text{Chow}(X)) = 0 .$$

where $\text{Chow}(\mathcal{X}_{z_i}, \omega_{\text{KE}}(\hat{z}_i))$ is the chow point corresponding to the Tian's embedding of \mathcal{X}_{z_i} with respect to the weak Kähler-Einstein metric $\omega_{\text{KE}}(\hat{z}_i)$.

Proof. Suppose this is not the case, there is a sequence $z_i = \text{Chow}(\mathcal{X}_{z_i}) \xrightarrow{i \rightarrow \infty} \text{Chow}(X)$ and

$$O_{z_j} \cap U = \emptyset \text{ with } O_{\hat{z}} := \text{SL}(N+1) \cdot \hat{z} .$$

In particular, by equipping each $\mathcal{X}_{\hat{z}_i}$ with a weak Kähler-Einstein metric $\omega_{\text{KE}}(\hat{z}_i)$, and taking the Gromov-Hausdorff limit Y , which is still embedded in \mathbb{P}^N by Lemma 8.3, we obtain

$$\text{Chow}(\mathcal{X}_{\hat{z}_i}, \omega_{\text{KE}}(\hat{z}_i)) \xrightarrow{i \rightarrow \infty} g \cdot \text{Chow}(Y) \in \overline{BO}_{\text{Chow}(X)} \setminus U \text{ for some } g \in \text{U}(N+1)$$

contradicting to the fact the limiting broken orbits \overline{BO}_z contains a unique K-polystable orbit. \square

Now we are ready to prove Theorem 8.5.

Proof of Theorem 8.5. Let U be the open set constructed above satisfying (56) and let

$$U_W^{\text{an}} = (U^{\text{ks}} \cap \mathbb{P}W) \times_{\mathbb{P}^{d,n;N}} Z^*.$$

After a possible shrinking, we may assume that all the points in U_W^{an} are GIT-semistable and every GIT-semistable point can be degenerated to a GIT-polystable point in U_W^{an} . Suppose $\text{Chow}(Y) \in U_W^{\text{an}}$ is GIT-polystable and strictly K-semistable. Then by Lemma 8.7, we can degenerate it to a variety $Y' \subset \mathbb{P}^N$ which is K-polystable such that

$$\text{Chow}(Y') \in U \cap \overline{\text{SL}(N+1) \cdot \text{Chow}(Y)} \subset Z^\circ \subset \mathbb{P}^{d,n;N},$$

and $\text{Chow}(Y')$ is *close* to $\text{Chow}(Y)$ in $\mathbb{P}^{d,n;N}$ in the sense that there is *short* (with respect to the metric $\text{dist}_{\mathbb{P}^{d,n;N}}$) path inside $\overline{\text{SL}(N+1) \cdot \text{Chow}(Y)}$ joining $\text{Chow}(Y)$ and $\text{Chow}(Y')$.

Using the transversality of the action of $\text{aut}(X)^\perp \subset \mathfrak{sl}(N+1)$ on $\mathbb{P}W \subset \mathbb{P}^{d,n;N}$, one can always find a $g \in \text{SL}(N+1)$ *close* to the identity such that

$$\text{Chow}(Y'') := g \cdot \text{Chow}(Y') \in \mathbb{P}W \times_{\mathbb{P}^{d,n;N}} Z^*,$$

where $Y'' \cong Y'$ is GIT-semistable. This allows us to find a *short* path inside $\overline{\text{SL}(N+1) \cdot \text{Chow}(Y)}$ joining $\text{Chow}(Y)$ and $\text{Chow}(Y'')$, which by transversality we may assume to be entirely contained in $\mathbb{P}W$ and satisfies $\text{Chow}(Y'') \in \overline{\text{Aut}(X) \cdot \text{Chow}(Y)}$. But this is absurd since $\text{Chow}(Y)$ is already GIT-polystable, no point on the boundary of $\overline{\text{Aut}(X) \cdot \text{Chow}(Y)}$ is semistable.

Conversely, suppose $\text{Chow}(Y) \in U_W^{\text{an}}$ and Y is K-polystable but $\text{Chow}(Y)$ is not GIT-polystable, then there is a 1-PS $\lambda \subset \text{Aut}(X)$ degenerating $\text{Chow}(Y)$ to a nearby GIT-polystable

$$\text{Chow}(Y') \in \overline{\text{Aut}(X) \cdot \text{Chow}(Y)} \cap U_W^{\text{an}}$$

by the classical GIT. Thus Y' is K-polystable by the previous paragraph, contradicting the assumption Y being K-polystable. Hence our proof is completed.

To pass from the analytic neighborhood to a Zariski neighborhood, we need to investigate the geometry of $\text{Aut}(X)$ -orbits. Let $U_W^{\text{ss}} \subset \mathbb{P}W$ containing $\text{Chow}(X)$ be the Zariski open set of *GIT-semistable points*. By [MFK94, Chapter 2, Proposition 2.14] and [Oda12, Lemma 2.11 and Lemma 2.12], we know that the set of GIT-polystable points in U_W^{ss} forms a *constructible* set. On the other hand, K-polystable points inside $U_W^{\text{ss}} \cap Z_{\text{red}}^\circ$ also form a constructible sets (see Remark 9.5) containing $\text{Chow}(X)$. These two constructible sets coincide along U_W^{an} after lifting to $\mathbb{P}W \times_{\mathbb{P}^{d,n;N}} Z^* \supset U_W^{\text{an}}$ by the proof above, so they must coincide on a Zariski open set.

Finally, we establish the last statement. By Theorem 1.1, the set of Chow points of the universal family \mathcal{X} of Kähler-Einstein \mathbb{Q} -Fano varieties obtained via Tian's embedding induces a $\text{U}(N+1)$ -invariant slice

$$(57) \quad \begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{P}^{d,n;N} = \mathbb{P}^M \\ & & \downarrow \\ & & S \end{array} \xrightarrow[\cong]{\text{homeomorphic}} \Sigma/\text{U}(N+1).$$

passing through z_0 . Moreover, the slice Σ satisfies the Assumption 9.6 thanks to the following:

Claim 8.8. Let X be a smoothable \mathbb{Q} -Fano variety admitting weak Kähler-Einstein metric. Then $\text{Aut}(X) = (\text{Isom}(X))^\mathbb{C}$. In particular, $\text{Aut}(X) = (\text{Aut}(X) \cap \text{U}(N+1))^\mathbb{C}$.

Proof. It follows from the proof of Theorem 4 in [CDS15c]. \square

By Theorem 9.9, we are able to construct an analytic open set $U_W \subset \mathbb{P}W \times_{\mathbb{P}^{d,n};N} Z^*$ that is stabilizer preserving. To obtain the Zariski openness, one only needs to observe the fact that

$$\text{Aut}(Z^*) := \{(z, G_z) \in Z^* \times \text{SL}(N+1) \mid G_z < \text{SL}(N+1) \text{ is the stabilizer of } z \text{ in } \text{SL}(N+1)\}$$

is a constructible set. Hence our proof is completed. \square

Remark 8.9. One notice that contrast to Theorem 8.5, there exists smooth Fano varieties admitting Kähler-Einstein metrics, which are not asymptotically Chow stable (see [OSY12]). On the other hand, Theorem 8.5 can be regarded as an extension of work [Sze10] in the case of smoothable \mathbb{Q} -Fano varieties.

Finally to prove Theorem 1.3, we need to show that for each \mathbb{C} -closed point $[z] \in [Z^*/\text{SL}(N+1)]$, $\overline{\{z\}}$ has a good moduli space in the sense of [AFS16, Proposition 3.1]. To to that, let us first establish the Assumption 9.10 in Section 9. Let $z = \text{Chow}(Y) \in U_W$ specializing to $z_0 = \overline{\text{Chow}(X)} \in U_W \subset \mathbb{P}^{d,n;N}$ via a 1-PS $\lambda(t) : \mathbb{G}_m \rightarrow \text{Aut}(X) < \text{SL}(N+1)$. Let $(\mathcal{Y} = \mathcal{X}|_C, X) \rightarrow (C = \overline{\lambda(t) \cdot z}, z_0) \subset U_W$ be the restriction of the universal family $\mathcal{X} \rightarrow Z^*$ to the pointed curve (C, z_0) and also we prefix a basis $\{s_i\} \subset \mathcal{O}_{\mathcal{Y}}(-rK_{\mathcal{Y}/C})$.

Lemma 8.10. *Under the notation introduced above, we have $\text{Aut}(Y) < \text{Aut}(X)$ for $z := \text{Chow}(Y)$ close to $z_0 = \text{Chow}(X)$.*

Proof. By property (3) in the proof of Lemma 3.1, for $z = \text{Chow}(Y) \in U_0$ we have $\text{aut}(Y) \subset \text{aut}(X)$, hence the identity component of $\text{Aut}(Y)$ lies in $\text{Aut}(X)$. We will assume from now on that $z = \text{Chow}(Y) \in \mathbb{P}W$ lies in a *small* analytic neighborhood of $z_0 = \text{Chow}(X) \in U_1$, i.e. z is *very close* to z_0 . This together with the fact that there always exists a *finite* subgroup $H < \text{Aut}(Y)$ that meets *every* connected component of $\text{Aut}(Y) < \text{SL}(N+1)$ imply that all we need is that: for any finite subgroup $H < \text{Aut}(Y)$, we have $H < \text{Aut}(X)$. To achieve that, let us choose H -invariant *smoothable* divisor $E \in |-mK_Y|^H$ so that $(Y, \frac{E}{m})$ is klt, the existence of such $E \subset Y$ is guaranteed by the following result.

Claim 8.11. Let Y be a *smoothable* \mathbb{Q} -Fano variety. Fix a finite group $H \subset \text{Aut}(Y)$. For m sufficiently divisible there is an invariant section $E \in |-mK_Y|^H$ such that $(Y, (1-\epsilon)E)$ is klt for any $0 < \epsilon \leq 1$ and *smoothable*. In particular, $(Y, \frac{1}{m}E)$ is *smoothable* and klt for $m > 1$.

Proof. Let $\mu : Y \rightarrow \tilde{Y}$ be the quotient of Y by H , and D be the branched divisor. So $\mu^*(K_{\tilde{Y}} + D) = K_Y$. In particular, (\tilde{Y}, D) is klt (, as klt is preserved under finite quotient [KM98, Theorem 5.20]) and $-(K_{\tilde{Y}} + D)$ is ample. Thus for a sufficiently divisible m satisfying $-m(K_{\tilde{Y}} + D)$ being very ample, we can choose a general section $F \in |-m(K_{\tilde{Y}} + D)|$ so that $(\tilde{Y}, D + (1-\epsilon)F)$ is klt for any $0 < \epsilon \leq 1$. Then $E := \mu^*(F)$ is H -invariant and $(Y, (1-\epsilon)E)$ is klt for any $0 < \epsilon \leq 1$. Finally, we justify that (Y, E) is actually *smoothable* as long as Y is. Since Y is a degeneration a smooth family $\{Y_t\}_t$, and every element in $|-mK_Y|$ can be represented as a degeneration of general members of $|-mK_{Y_t}|$, from which we conclude (Y, E) is a degeneration of smooth pairs $\{(Y_t, E_t)\}_t$. \square

Then by Theorem 1.2 and 5.2, $(Y, \frac{E}{m})$ admits a continuous family of Kähler metric $\{\omega_Y(\beta)\}$ solving

$$\text{Ric}(\omega_Y(\beta)) = \beta\omega_Y(\beta) + \frac{1-\beta}{m}[E] \text{ on } Y,$$

from which we obtain

$$(58) \quad \text{Chow}(Y, \omega_Y(\beta)) \xrightarrow{\beta \rightarrow 1} \text{U}(N+1) \cdot \text{Chow}(X) \subset \mathbb{P}^{d,n;N}$$

thanks to Theorem 7.2, where $\text{Chow}(Y, \omega_Y(\beta))$ is the Chow point corresponding to the Tian's embedding of $Y \subset \mathbb{P}^N$ with respect to the metric $\omega_Y(\beta)$ on $Y \subset \mathbb{P}^N$ and any prefixed basis $\{s_i\} \subset H^0(\mathcal{O}_Y(-rK_Y))$. This allows us to introduce a *continuous* family of Hermitian metric $h_{\text{KE}}(\beta(t))$ with $\beta(t) := 1 - |t|$ on $\mathcal{O}_{\mathcal{Y}_t}(-K_{\mathcal{Y}_t}) \rightarrow \mathcal{Y}_t$ for $0 < |t| := \text{dist}_C(t, 0) < 1$, such that $\omega_Y(\beta(t)) = -\sqrt{-1}\partial\bar{\partial} \log h_{\text{KE}}(\beta)$. By (58), the metric $h_{\text{KE}}(\beta(t))$ can be continuously extended to $0 \in C$. Now let $\{s_i\}$ be the local basis of $\pi_*\mathcal{O}_{\mathcal{Y}}(-rK_{\mathcal{Y}/C})|_{\{|t|<1\} \subset C} = \pi_*(\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{Y}})$ corresponding to the coordinate sections of $\mathcal{O}_{\mathbb{P}^N}(1)$ such that $\{s_i(0)\}$ induces Tian's embedding for $z_0 = \text{Chow}(X)$ and define

$$A_{\text{KE}}(t, \beta(t)) = [(s_i, s_j)_{\text{KE}, \beta(t)}(t)]$$

with

$$(s_i, s_j)_{\text{KE}, \beta(t)} = \int_{\mathcal{Y}_t} \langle s_i(t), s_j(t) \rangle_{h_{\text{KE}}^{\otimes r}(\beta(t))} \omega_Y^n(\beta(t)),$$

then we obtain a family of Tian's embedding

$$(59) \quad T : (\mathcal{Y}_t, \mathcal{E}_t; \omega_Y(\beta(t))) \longrightarrow \mathbb{P}^N \text{ with } (\mathcal{Y}_t, \mathcal{E}_t) \cong (Y, E) \text{ for } t \neq 0.$$

given by $\{g(t) \circ s_j(t)\}_{j=0}^N$ with $g(t) = A_{\text{KE}}^{-1/2}(\beta(t))$. The map T extends to $\mathcal{Y}_0 = X$ thanks to the continuity of the metric $h_{\text{KE}}(\beta(t))$ at $0 \in C$.

Now by our choice of z_0 and basis $\{s_i(t)\}$, we have $A_{\text{KE}}(0, 1) = I_{N+1} \in \text{SL}(N+1)$, and hence

$$(60) \quad g(t) \sim I_{N+1} + O(t).$$

This implies that

$$(61) \quad \tilde{z}(t) := \text{Chow}(\mathcal{Y}_t, \omega_Y(\beta(t))) = g(t) \cdot z_t \in U_{1,\epsilon} := \exp(\mathfrak{aut}(X)_{<\epsilon}^\perp) \cdot U_1$$

for $0 < |t| \ll 1$, where $z_t = \lambda(t) \cdot \text{Chow}(Y)$. Since $\omega_Y(\beta(t))$ is a conical Kähler-Einstein metric on \mathcal{Y}_t , it follows from the log version of Claim 8.8 (cf. [CDS15c, Theorem 4]) that

$$H_{\tilde{z}(t)} = g(t) \cdot H_{z(t)} \cdot g(t)^{-1} < U(N+1) \text{ where } H_{z(t)} = \lambda(t) \cdot H \cdot \lambda(t)^{-1}.$$

By Lemma 9.8 and (60), we obtain that $H_{z(t)} < \text{Aut}(X)$ and hence $H < \text{Aut}(X)$ as $\lambda(t) < \text{Aut}(X)$ by our choice. On the other hand, by transversality of $\mathfrak{aut}(X)^\perp$ -action on $U_{1,\epsilon}$, for $0 < t \ll 1$ we have $\text{Aut}_0(Y) < \text{Aut}(X)$ where $\text{Aut}_0(Y)$ is the identity component of $\text{Aut}(Y)$. This implies that $\text{Aut}(Y) = \langle \text{Aut}_0(Y), H \rangle < \text{Aut}(X)$ where $\langle \text{Aut}(Y), H \rangle$ is the subgroup generated by H and $\text{Aut}_0(Y)$ and our proof is completed. \square

As a direct consequence of Lemma 8.3, 8.10 and 8.7, we have the following:

Corollary 8.12. *After a possible shrinking of the Zariski open neighborhood $z_0 \in U_W \subset \mathbb{P}W \times_{\mathbb{P}^d, n; N} Z^*$, we have*

$$\text{SL}(N+1)_z < \text{Aut}(X), \quad \forall z \in U_W$$

where $\text{SL}(N+1)_z$ is the stabilizer of z inside $\text{SL}(N+1)$. In particular, Assumption 9.10 holds in this case.

Next in order to apply Lemma 9.14 in Section 9, we now establish Assumption 9.13. Let us fix $G = \text{SL}(N+1)$ and $G_{z_0} = \text{Aut}(X)$.

Lemma 8.13. *Let $z_0 \in U_r \subset \mathbb{P}W$ be defined in Definition 9.12 and*

$$U_{Z^*, r} := U_r \times_{\mathbb{P}^d, n; N} Z^*.$$

Then for $0 < r$ sufficient small, we have $U_{Z^, r} \subset U^{\text{fd}}$, i.e. Assumption 9.13 is satisfied on $U_{Z^*, r}$.*

Proof. In order to better illustrate the idea, let us first deal with the case that z_0 is K-stable, hence $G_{z_0} < \infty$. As we have seen in the proof of Theorem 8.5, there is a continuous $U(N+1)$ -invariant slice $z_0 \in \Sigma \subset \mathbb{P}^{d, n; N}$ obtained via Tian's embedding. By the continuity of Σ and transversality of the $\mathfrak{g}_{z_0}^\perp$ -action on U_0 (cf. the proof Lemma 3.1), for some $0 < r'' < r' \ll 1$ and $0 < \epsilon \ll 1$ we have

$$(62) \quad B_{Z^*}(z_0, r'') \subset U_{r'} \cap \exp \mathfrak{g}_{z_0, < \epsilon}^\perp \cdot \Sigma,$$

where $\mathfrak{g}_{z_0, < \epsilon}^\perp := \{\xi \in \mathfrak{g}_{z_0} \mid |\xi| < \epsilon\}$ and $B_{Z^*}(z_0, r'')$ denotes the ball of radius ϵ centered at $z_0 \in Z^*$ with respect to a prefixed continuous metric on Z^* . Moreover, by choosing a small r if necessary, we may assume \mathcal{X}_z is K-stable for all $z \in B_{Z^*}(z_0, r'')$. To see the lemma, let $\{s_i\}$ be the local basis of $\pi_*(\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}})$ corresponding to the coordinate sections of \mathbb{P}^N such that the induced embedding of $X = \mathcal{X}_{z_0} \subset \mathbb{P}^N$ gives rise to $\text{Chow}(X)$. Now let us equip the line bundle $\mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/Z^*, \text{kps}}) \cong \mathcal{O}_{\mathbb{P}^N}(1)|_{\mathcal{X}}$ with a Hermitian metric which gives rise to the *unique* Kähler-Einstein metric when restricted to each \mathcal{X}_z with $z \in B_{Z^*}(z_0, r'')$, and we can introduce the matrix $A_{\text{KE}}(z)$ as in the proof of Lemma 8.10. Then (62) follows from the continuity of $A_{\text{KE}}(z)$ with respect to $z \in Z^*$ and $A_{\text{KE}}(z_0) = I_{N+1}$ (as $X \subset \mathbb{P}^N$ is a Tian's embedding).

As a consequence, for any pair $(z, g) \in B_{Z^*}(z_0, r'') \times G$ satisfying $g \cdot z \in B_{Z^*}(z_0, r'')$, there are $h', h'' \in G$ such that under the quotient map

$$[\cdot] : G \rightarrow G/G_{z_0},$$

$[h'], [h''] \in G/G_{z_0}$ are perturbations of $[1] \in G/G_{z_0}$ and $h' \cdot z, h'' \cdot g \cdot z \in \Sigma$. Since both $h \cdot z$ and $h' \cdot g \cdot z$ are the Chow points of Tian's embedding of the *same* \mathbb{Q} -Fano variety, we know that $u := h'^{-1} \cdot h'' \cdot g \in U(N+1)$. This implies that $g \cdot z = h \cdot z$ with $h = h''^{-1} \cdot h' \cdot u$ and $[h]$ being *uniformly bounded* (which depend only on $B_{Z^*}(z_0, r'')$ and z_0) in G/G_{z_0} . Since the property whether or not z lies in U^{fd} is independent of the G_{z_0} -translation, we conclude that Assumption 9.13 holds for K-polystable points lies in $U_{Z^*, r} \subset G_{z_0} \cdot B_{Z^*}(z_0, r'')$ for some $0 < r < r''$.

For the general case, let us introduce a general divisor $\mathcal{D} \in |-mK_{\mathcal{X}}|$ for sufficiently divisible m such that

- (1) $(\mathcal{X}, \mathcal{D})|_{U_W}$ (, where U_W is given in the proof of Theorem 8.5) are family of \mathbb{Q} -Fano variety;
- (2) \mathcal{D}_z is smooth whenever \mathcal{X}_z is for $z \in U_W$.

Then by Theorem 1.2 we can construct a *thickened* (due to the introducing of \mathcal{D}) continuous $U(N+1)$ -invariant slice $\Sigma_{\frac{1-\beta}{m}\mathcal{D}} \subset \mathbb{P}^{d, n; N}$ using Tian's embedding of $\mathcal{X}_z \subset \mathbb{P}^N$ with respect to the *unique* conical Kähler-Einstein metric

$$\text{Ric}(\omega_{\mathcal{X}_z}(\beta)) = \beta \omega_{\mathcal{X}_z}(\beta) + \frac{1-\beta}{m} [\mathcal{D}_z] \text{ on } \mathcal{X}_z$$

for all $z \in U_W$ near z_0 . In particular, Theorem 1.2 and 7.2 imply that $\Sigma_{\frac{1-\beta}{m}\mathcal{D}} \rightarrow \Sigma$ in the sense that $\forall \epsilon > 0$, $\Sigma_{\frac{1-\beta}{m}\mathcal{D}}$ falls into a ϵ -tubular neighborhood of Σ as $\beta \rightarrow 1$. This implies that for $0 < r' \ll 1$ and $z, z' \in B_{Z^*}(z_0, r')$ that are contained in

$$(G \cdot \text{Chow}(\mathcal{X}_z)) \cap (U(N+1) \cdot \exp \sqrt{-1} \mathfrak{g}_{z_0, < \epsilon}^\perp \cdot B_{Z^*}(z_0, r')) \text{ (cf. (70))}$$

with $\mathfrak{g}_{z_0, < \epsilon}^\perp := \{\xi \in \mathfrak{g}_{z_0} \mid |\xi| < \epsilon\}$, the $U(N+1)$ -orbits for Tian's embedding of $(\mathcal{X}_z, \mathcal{D}_z)$ and $(\mathcal{X}_{z'}, \mathcal{D}_{z'})$ are very close in the sense that they can be translated to each other by an element $h \in U(N+1) \cdot \exp \sqrt{-1} \mathfrak{g}_{z_0, < \epsilon}^\perp \cdot G_{z_0} \subset G$ (i.e. $[h] \in G/G_{z_0}$ is bounded in the sense of (70)). In particular, this allows us to treat these two $U(N+1)$ -orbits as *almost identical* one and we argue exactly the same way as the K-stable case. This completes the justification of Assumption 9.13 for a neighborhood of $z_0 \in U_{Z^*, r}$ for some sufficient small $r > 0$. \square

Proof of Theorem 1.3. By [AFS16, Proposition 3.1], proving our statement boils down to establishing the following: for any \mathbb{C} -closed point $[z_0] \in [Z^*/\text{SL}(N+1)]$ there is an *affine* neighbourhood $z_0 \in U_W \subset \mathbb{P}W$ determined in (53) such that

- (1) The morphism $[U_W/G_{z_0}] \rightarrow [Z^*/G]$ is *affine and strongly étale* (i.e. stabilizer preserving and sending closed point to closed point), and
- (2) For any $z \in Z^*$ specializing to z_0 under G -action, the closure of substack $[z]$ inside $[Z^*/G]$, $\overline{\{[z]\}} \subset [Z^*/G]$ admits a good moduli space.

Here we fix $G = \text{SL}(N+1)$ and $G_{z_0} = \text{Aut}(X)$.

We have shown the morphism is strongly étale by Theorem 8.5. Next we confirm the affineness. Since $Z^* \rightarrow [Z^*/\text{SL}(N+1)]$ is faithfully flat, it suffices to show that

$$\phi: G \times_{G_{z_0}} U_W \rightarrow Z^*$$

is affine. Since ϕ is quasi-finite and Z^* is separated, it suffices to choose U_W such that $G \times_{G_{z_0}} U_W$ is affine. Let $U_W \subset Z^*$ be an G_{z_0} -invariant affine open set then the affineness of $G \times_{G_{z_0}} U_W$ follows from the fact that

$$(G \times_{G_{z_0}} U_W) // G \cong U_W // G_{z_0}$$

and $G \times_{G_{z_0}} U_W$ is the inverse image of the affine neighborhood

$$\pi_W|_{U_W}(z_0) = 0 \in U_W // G_{z_0} \text{ with } \pi_W \text{ defined in (69)}$$

under the GIT quotient by G .

Now we establish the second condition. Since we have already established the *uniqueness* of minimal orbit contained in \overline{BO}_{z_0} stated after diagram (55), all we need is the affineness of $G \cdot \pi_W^{-1}(0)$ as it implies that for any $z \in Z^*$ satisfying $\overline{G} \cdot z \ni z_0$ the closure of $[z] \in [Z^*/G]$ is a closed substack of $[G \cdot \pi_W^{-1}(0)/G]$, which can be written as the form $[\text{Spec}(A)/G]$ for some affine scheme $\text{Spec}(A)$, hence $\overline{[z]}$ admits a good moduli space.

To obtain the affineness, one notices that Theorem 8.5 and Corollary 8.12 guarantee the Assumption 9.10, also we have already established Assumption 9.13 by Lemma 8.13. Thus the morphism

$G \times_{G_{z_0}} U_r \rightarrow G \cdot U_r$ (cf. Definition 9.12) is a *finite* morphism for $0 < r \ll 1$ by Lemma 9.14 in Section 9. By restricting to the fiber over $[z_0] \in [Z^*/G]$, we have a finite morphism

$$G \times_{G_{z_0}} \pi_W^{-1}(0) \longrightarrow G \cdot \pi_W^{-1}(0) .$$

Since $G \times_{G_{z_0}} \pi_W^{-1}(0)$ is a fiber of a GIT quotient morphism, we conclude that $G \cdot \pi_W^{-1}(0)$ is affine.

Finally to prove the last statement of Theorem 1.3, we observe that Lemma 8.3 implies that the closed points of \mathcal{KF}_N stabilizes. However, since \mathcal{KF}_N is semi-normal, we indeed know that they are isomorphic (see [Kol96, 7.2]). \square

Remark 8.14. We remark that if we work with the Hilbert scheme instead of the Chow then there is no need for us to take the semi-normalization to guarantee the existence of universal family, and we can take the local GIT quotient of a similarly defined Z_{red}° for each N . Although we cannot conclude that those GIT quotients will be stabilized for $N \gg 1$, their semi-normalizations indeed will be.

Another reason we work over a seminormal base is that the condition of being smoothable does not yield a reasonable moduli functor for schemes, e.g., in general there is no good definition of smoothable varieties over an Artinian ring.

9. APPENDIX

9.1. Constructibility of kst. In this section, we will prove Proposition 7.5 in a more general setting. First, let us recall some basics from [MFK94, section 2 of Chapter 2]. Let G be a reductive group acting on a projective variety (Z, L) polarised by a G -linearized very ample line bundle L .

Definition 9.1. The *rational flag complex* $\Delta(G)$ is the set of non-trivial 1-PS's λ of G modulo the *equivalence* relation: $\lambda_1 \sim \lambda_2$ if there are positive integers n_1 and n_2 and a point $\gamma \in P(\lambda_1)$ such that

$$\lambda_2(t^{n_2}) = \gamma^{-1} \lambda_1(t^{n_1}) \gamma \text{ for all } t \in \mathbb{G}_m$$

where

$$P(\lambda) := \left\{ \gamma \in G \mid \lim_{t \rightarrow 0} \lambda(t) \gamma \lambda(t^{-1}) \text{ exists} \right\} \subset G$$

is the unique *parabolic subgroup* associated to λ . The point of $\Delta(G)$ defined by λ will be denoted by $\Delta(\lambda)$. In particular, for a maximal torus $T \subset G$, $\Delta(T) = \text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$.

Then we have the following

Lemma 9.2 (Chapter 2, Proposition 2.7, [MFK94]). *For any 1-PS $\lambda : \mathbb{G}_m \rightarrow G$, let $\mu^L(z, \lambda)$ denote the λ -weight of $z \in Z$ with respect to the G -linearization of L . Then for any $(\gamma, z) \in G \times Z$, we have*

$$\mu^L(z, \lambda) = \mu^L(\gamma z, \gamma \lambda \gamma^{-1}) .$$

Moreover, if $\gamma \in P(\lambda)$ then $\mu^L(z, \lambda) = \mu^L(z, \gamma \lambda \gamma^{-1})$.

and a slight extension of [MFK94, Chapter 2, Proposition 2.14] and [Oda12, Lemma 2.11], whose proof will be omitted

Lemma 9.3. *Let $T \subset G$ be a maximal torus and L, M be two G -linearized ample line bundles over Z . Then there is a finite set of linear functional $l_1^L, \dots, l_{r_L}^L, l_1^M, \dots, l_{r_M}^M$ which are rational on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ with the following property:*

$$(63) \quad \forall z \in Z, \exists I(z, L) \subset \{1, \dots, r_L\}, I(z, M) \subset \{1, \dots, r_M\}$$

such that the λ -weight of $z \in Z$ with respect to the linearization of G on $L \otimes M^{-1}$ is given by

$$\mu^L(z, \lambda) - \mu^M(z, \lambda) = \max\{l_i^L(\lambda) \mid i \in I(z, L)\} - \max\{l_i^M(\lambda) \mid i \in I(z, M)\}$$

for all 1-PS $\lambda \subset T$. Moreover, the function

$$\begin{aligned} \psi^{L, M} : Z &\longrightarrow 2^{\{1, \dots, r_L\}} \sqcup 2^{\{1, \dots, r_M\}} \\ z &\longmapsto I(z; L, M) := I(z, L) \sqcup I(z, M) \end{aligned}$$

are constructible in the sense that $\forall I \in 2^{\{1, \dots, r_L\}} \sqcup 2^{\{1, \dots, r_M\}}$, the set $\psi^{-1}(I) \subset Z$ is constructible.

For any line bundle that can be written as $L \otimes M^{-1}$ with both L and M being G -linearized and very ample, we can similarly show that Z can be decomposed into a union of finitely many constructible sets indexed by $2^{\{1, \dots, r_{L_1}\}} \sqcup 2^{\{1, \dots, r_{L_2}\}}$, such that restricted on each piece,

$$\mu^{L \otimes M^{-1}}(z, \lambda) = \mu^L(z, \lambda) - \mu^M(z, \lambda)$$

is a rational function on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$.

Proposition 9.4. *Let G act on an polarized variety (Z, L) . Let $M_i, i = 1, 2$ be two G -linearized line bundles on Z (not necessarily being ample). For $z \in Z$ and $\delta \in \Delta(G)$, we define*

$$\nu_{1-\beta}^{M_1, M_2}(z, \delta) := \frac{\mu^{M_1}(z, \lambda) - (1 - \beta)\mu^{M_2}(z, \lambda)}{|\lambda|} \text{ with } \Delta(\lambda) = \delta$$

and

$$\varpi_G^{M_1, M_2}(z) := \sup \left\{ \beta \in (0, 1] \mid \inf_{\delta \in \Delta(G)} \nu_{1-\beta'}^{L, M}(z, \delta) \geq 0, \forall \beta' \in [0, \beta) \right\}$$

or 0 if the right hand side is an empty set. Suppose $S \subset Z$ is a constructible set such that $\varpi_G^{M_1, M_2}|_S > 0$. Then $\varpi(M_1, M_2)$ defines a \mathbb{Q} -valued constructible function on S , i.e. $S = \sqcup_i S_i$ is a union of finite constructible sets with $\varpi(M_1, M_2)$ being constant on each S_i .

Proof. We replace $L \rightarrow Z$ by its power such that $L_1 := L \otimes M_1$ and $L_2 := L \otimes M_2$ are both ample. Then we fix a maximal torus $T \subset G$ and let $\{l_i^{L_1}\}$ and $\{l_i^{L_2}\}$ be the *rational* linear functionals on $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ associated to $L_i, i = 1, 2$. By Lemma 9.3, for any $I \in 2^{\{1, \dots, r_{L_1}\}} \sqcup 2^{\{1, \dots, r_{L_2}\}}$, $S_I^T := \psi^{-1}(I) \cap S$ is a constructible set. Now we define

$$\varpi_T^{M_1, M_2}(z) := \sup \left\{ \beta \in (0, 1] \mid \inf_{\delta \in \Delta(T)} \nu_{1-\beta'}^{L, M}(z, \delta) \geq 0, \forall \beta' \in [0, \beta) \right\}$$

or 0 if the right hand side is an empty set. In other words, it is the *first time* such that the difference of two *rational* piecewise linear *convex* functions

$$\mu^{L_1}(z, \cdot) - (1 - \beta)\mu^{L_2}(z, \cdot) - \beta\mu^L(z, \cdot) = \mu^{M_1}(z, \cdot) - (1 - \beta)\mu^{M_2}(z, \cdot)$$

vanishes along a ray in $\text{Hom}_{\mathbb{Q}}(\mathbb{G}_m, T)$ or in $\{0, 1\}$. Clearly, we have $\beta_I \in \mathbb{Q}$ and they are independent of the choice of L .

Now in order to pass from $\varpi_T^{M_1, M_2}$ to $\varpi_G^{M_1, M_2}$, let us recall Chevalley's Lemma [Har77, Chapter II, Exercise 3.19] which states that the image of constructible set under an algebro-geometric morphism is again constructible. By applying it to the group action morphism

$$G \times Z \longrightarrow Z,$$

we obtain that $S_I^G := G \cdot (\psi^{-1}(I) \cap S) \supset S_I^T$ are all constructible $\forall I \in 2^{\{1, \dots, r_{L_1}\}} \sqcup 2^{\{1, \dots, r_{L_2}\}}$. Now for any 1-PS λ , there is a $\gamma \in G$ such that $\gamma\lambda\gamma^{-1} \subset T$. By Lemma 9.2, we have $\mu^{L_i}(z, \lambda) = \mu^{L_i}(\gamma z, \gamma\lambda\gamma^{-1})$, $i = 1, 2$, which implies that

$$\varpi_G^{M_1, M_2}(z) = \min \left\{ \beta_J \mid S_J^T \cap G \cdot z \neq \emptyset \text{ for } J \in 2^{\{1, \dots, r_{L_1}\}} \sqcup 2^{\{1, \dots, r_{L_2}\}} \right\}$$

To see it is a constructible function on the constructible set $G \cdot S$, one notices that all possible finite intersections of $\{S_J^G\}_J$ form a stratification of $G \cdot S$ into constructible sets and $\varpi_G^{M_1, M_2}$ is constant on each stratum. □

Now to apply the above set up to the β -K-stability of $(X, D) \subset \mathbb{P}^N$ with respect to the $\text{SL}(N+1)$ action. Let $N+1 = \dim H^0(X, K_X^{\otimes(-r)})$ and we define

$$(64) \quad Z := \left\{ \text{Chow}(X, D) \mid \begin{array}{l} (X, D) \subset \mathbb{P}^N \times \mathbb{P}^N \text{ be a klt pair satisfying:} \\ D \subset X, \deg(X, D) = (d, \delta) \text{ and } \mathcal{O}_{\mathbb{P}^N}(1)|_X \cong K_X^{-\otimes r}. \end{array} \right\} \subset \mathbb{P}^{d, n; N}$$

to be the Chow variety of log \mathbb{Q} -Fano $(X, D) \subset \mathbb{P}^N$, where $d = (-rK_X)^n$, $\delta = md/r$ and $\mathbb{P}^{d, n; N} := \mathbb{P}^{d, n; N} \times \mathbb{P}^{\delta, n-1; N}$. Let $\Lambda_{\text{CM}} \rightarrow Z$ (cf. [FR06, Definition 2.3] or [PT06, equation (2.4)]) be the CM-line bundle over Z .

Proof of Proposition 7.5. Let us introduce

$$M_1 := \Lambda_{\text{CM}} \text{ and } M_2 := \mathcal{O}_{\mathbb{P}^d, n}(-1)^{\otimes \frac{1-\beta}{mr^n}} \otimes \mathcal{O}_{\mathbb{P}^d, n-1}(1)^{\otimes \frac{1-\beta}{r^{n+1}}} \text{ (cf. (2)).}$$

By Theorem 5.2, we know $(\mathcal{X}_t, \mathcal{D}_t)$ is β -K-stable $\forall t \in C$ and $\beta \in (0, \beta_0]$. After removing finite number of points from C , we obtain a quasiprojective $0 \in S \subset C$ over which $\pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})|_S \cong \mathcal{O}_S^{\oplus N+1}$. By fixing a basis of $\pi_* \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})|_S$, we obtain an embedding

$$\iota : (\mathcal{X}, \mathcal{D}; \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/C})) \times_C S \longrightarrow \mathbb{P}^N \times \mathbb{P}^N \times S$$

which in turn induces an embedding $S \subset Z$ with S being constructible and $\varpi_{\text{SL}(N+1)}^{M_1, M_2} \geq \beta_0 > 0$. By applying Proposition 9.4 to $S \subset Z$, we obtain $\text{kst}(\mathcal{X}_t, \mathcal{D}_t) = \varpi_{\text{SL}(N+1)}^{M_1, M_2}(t)$, $\forall t \in S$ is a constructible function. Our proof is completed. \square

Remark 9.5. It was first observed in [Oda12] (also see [Pau12]) that we can also conclude from the argument above that the K-polystable locus in S is also constructible.

9.2. Stabilizer Preserving Property.

9.2.1. *Richardson's example.* Consider $\text{SL}(2)$ -action on

$$\text{Sym}^{\otimes 3} \mathbb{C}^2 = H^0(\mathcal{O}_{\mathbb{P}^1}(3)) = \text{Span}_{\mathbb{C}}\{X^3, X^2Y, XY^2, Y^3\}$$

induced by the standard action on \mathbb{C}^2 . Then the stabilizer of $p_0(X, Y) = (X - Y)(X + Y)^2$ is trivial and the stabilizer of $p(X, Y) = (X - Y)(X - \omega Y)(X - \omega^2 Y)$ is given by

$$\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \in \text{SL}(2, \mathbb{C}) \text{ with } \omega^3 = 1.$$

Let

$$\alpha(t) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} t^2 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in \text{GL}(2, \mathbb{C}) = \frac{1}{2} \begin{bmatrix} t^2 + 1/t & -t^2 + 1/t \\ -t^2 + 1/t & t^2 + 1/t \end{bmatrix} \in \text{GL}(2, \mathbb{C})$$

then

$$\alpha(t) : \begin{cases} X - Y \longrightarrow t^2(X - Y) \\ X + Y \longrightarrow t^{-1}(X + Y) \end{cases}$$

hence fixes $p_0(X, Y) = \frac{3}{4}(X - Y)(X + Y)^2 \in \text{Sym}^{\otimes 3} \mathbb{C}^2$. Now let us define

$$\begin{aligned} p_t(X, Y) &= p(\alpha(t) \cdot X, \alpha(t) \cdot Y) \\ &= \frac{1}{4} t^2 (X - Y)(t^2(1 + \omega) + t^{-1}(1 - \omega))X + (-t^2(1 + \omega) + t^{-1}(1 - \omega))Y \cdot \\ &\quad \cdot (t^2(1 + \omega^2) + t^{-1}(1 - \omega^2))X + (-t^2(1 + \omega^2) + t^{-1}(1 - \omega^2))Y \\ &= \frac{1}{4} (X - Y)(t^3(1 + \omega) + (1 - \omega))X + (-t^3(1 + \omega) + (1 - \omega))Y \cdot \\ &\quad \cdot (t^3(1 + \omega^2) + (1 - \omega^2))X + (-t^3(1 + \omega^2) + (1 - \omega^2))Y, \end{aligned}$$

then we have

$$\lim_{t \rightarrow 0} p_t(X, Y) = \frac{3}{4}(X - Y)(X + Y)^2$$

and the stabilizer of p_t is the subgroup $\langle \zeta_t := \zeta_{p_t} \rangle \subset \text{SL}(2)$ with

$$\zeta_{p_t} := \alpha(t^{-1}) \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \alpha(t) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \omega + \omega^{-1} & t^{-3}(\omega - \omega^{-1}) \\ t^3(\omega - \omega^{-1}) & \omega + \omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{t \rightarrow 0} \infty,$$

In particular, the family of stabilizers $\langle \zeta_t \rangle \subset \text{SL}(2, \mathbb{C})$ is *unbounded* as $t \rightarrow 0$ unless $\omega = 1$.

9.2.2. Main results. Before we precede to the proof, let us collect some basic facts on compact Lie groups acting on \mathbb{P}^M . Let K be a compact Lie group and $\rho : K \rightarrow \mathrm{SU}(M+1)$ be a linear representation and $\rho^\mathbb{C} : G = K^\mathbb{C} \rightarrow \mathrm{SL}(M+1)$ be its complexification. Let $z_0 \in \mathbb{P}^N$ with stabilizer $G_{z_0} = (K_{z_0})^\mathbb{C} := (G_{z_0} \cap K)^\mathbb{C}$. Let $\mathfrak{k}_{z_0} = \mathrm{Lie}(K_{z_0})$ be the Lie algebra and we fix a bi-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on \mathfrak{k} and let $\mathfrak{k}_{z_0}^\perp \subset \mathfrak{k}$ be its orthogonal complement with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$. Then the infinitesimal action $\sigma_{z_0} : \mathfrak{g} \rightarrow T_{z_0}\mathbb{P}^M$ is G_{z_0} -equivariant in the sense that

$$\sigma_{z_0}(\mathrm{Ad}_g \xi) = g \cdot \sigma_{z_0}(\xi) \text{ for all } g \in G_{z_0},$$

and there is a G_{z_0} -invariant linear subspace $z_0 \in \mathbb{P}W \subset \mathbb{P}^M$ such that

$$(65) \quad \mathbb{P}^M = \mathbb{P}(W \oplus \mathbb{C}z_0 \oplus (\mathfrak{k}_{z_0}^\perp)^\mathbb{C}) \text{ with } (\mathfrak{k}_{z_0}^\perp)^\mathbb{C} := \mathfrak{k}_{z_0}^\perp \otimes \mathbb{C},$$

where $W \oplus \mathbb{C}z_0 \oplus (\mathfrak{k}_{z_0}^\perp)^\mathbb{C}$ is a decomposition as G_{z_0} -invariant subspaces.

Assumption 9.6 (Boundedness). There is a *continuous* K -invariant slice $\Sigma \subset \mathbb{P}^M$ containing z_0 , such that $G_z = (G_z \cap K)^\mathbb{C}$ for all $z \in \Sigma$.

Consider the map

$$(66) \quad \begin{array}{ccc} G \times \mathbb{P}W & \xrightarrow{\phi} & G \cdot \mathbb{P}W \subset \mathbb{P}^M \\ (g, w) & \mapsto & g \cdot w \end{array}$$

then for $\xi \in \mathfrak{g}_{z_0}$ and $\delta w \in T_{z_0}\mathbb{P}W$ we have

$$d\phi|_{(e, z_0)}(\xi, \delta w) = \sigma_{z_0}(\xi) + \delta w \in T_{z_0}\mathbb{P}^N \cong T_{z_0}\mathbb{P}W \oplus (\mathfrak{k}_{z_0}^\perp)^\mathbb{C},$$

and as a consequence $\ker d\phi|_{(1, z_0)} = \mathfrak{g}_{z_0}$. Now let us define an open set

$$U_0 := \left\{ w \in \mathbb{P}W \mid \mathrm{rk} \left(q \circ d\phi|_{\{1\} \times \mathbb{P}W} : \mathfrak{g} \times T\mathbb{P}W \rightarrow (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W \right) = \dim \mathfrak{g}_{z_0}^\perp \right\} \subset \mathbb{P}W$$

where $q : T\mathbb{P}^N|_{\mathbb{P}W} \rightarrow (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W$ is the quotient map. Then we have

Lemma 9.7. $U_0 \subset \mathbb{P}W$ is a G_{z_0} -invariant Zariski open set.

Proof. Note that the Zariski openness follows from the fact that $q \circ d\phi \in H^0(\mathbb{P}W, T(G \times \mathbb{P}W)|_{\{1\} \times \mathbb{P}W} \otimes (T\mathbb{P}^N|_{\mathbb{P}W})/T\mathbb{P}W)$. So all we need is the G_{z_0} -invariance. To achieve that, one notices that for any $g \in G_{z_0}$, $\xi \in \mathfrak{g}$ and $w \in \mathbb{P}W$ we have

$$(g \cdot)_* \sigma_w(\xi) = \sigma_{g \cdot w}(\mathrm{Ad}_g \xi),$$

which implies that

$$\sigma_w(\xi) \in T_w\mathbb{P}W \iff \sigma_{g \cdot w}(\mathrm{Ad}_g \xi) \in T_{g \cdot w}\mathbb{P}W.$$

Now $w \in U_0$ can be characterized as $q \circ d\phi$ being of full rank which is also equivalent to

$$(67) \quad \sigma_w(\xi) \in T_w\mathbb{P}W \iff \xi \in \mathfrak{g}_{z_0}.$$

If $g \cdot w \notin U_0$ then there is a $0 \neq \mathrm{Ad}_g \xi \in \mathfrak{g}_{z_0}^\perp$ such that $\sigma_{g \cdot w}(\mathrm{Ad}_g \xi) \in T_{g \cdot w}\mathbb{P}W$, and hence $\sigma_w(\xi) \in T_w\mathbb{P}W$. On the other hand, we have decomposition $\mathfrak{g} = \mathfrak{g}_{z_0} \oplus \mathfrak{g}_{z_0}^\perp$ as a G_{z_0} -module via the Adjoint action thanks to the *reductivity* of G_{z_0} . This implies that $0 \neq \xi \in \mathfrak{g}_{z_0}^\perp$, contradicting to (67) and the assumption that $w \in U_0$. Thus our proof is completed. \square

Now ϕ is G_{z_0} -invariant with respect to the action $h \cdot (g, w) = (gh^{-1}, h \cdot w)$, hence it descends to a K -invariant map, which by abusing of notation is still denoted by

$$\begin{array}{ccc} G \times_{G_{z_0}} \mathbb{P}W & \xrightarrow{\phi} & G \cdot \mathbb{P}W \subset \mathbb{P}^M \\ (g, w) & \mapsto & g \cdot w \end{array}$$

Moreover, it is a *bi-holomorphism* (see the proof of [Sja95, Theorem 1.12]) from a K -invariant tubular neighborhood

$$(68) \quad U_\epsilon := \left\{ (g \exp \sqrt{-1}\xi, w) \in G \times_{G_{z_0}} V \mid g \in K, \xi \in \mathfrak{k}_{<\epsilon} \right\} \text{ with } \mathfrak{k}_{<\epsilon} := \{\xi \in \mathfrak{k} \mid |\xi| < \epsilon\}$$

of the orbit $K \cdot z_0 \cong K/K_{z_0}$ onto $\phi(U_\epsilon) = K \cdot \exp \mathfrak{k}_{<\epsilon} \cdot V$ for $0 < \epsilon \ll 1$, where $z_0 \in V \subset \mathbb{P}W$ is a K -invariant *analytic* open neighborhood.

Now suppose $\tilde{g} = g \cdot \exp \sqrt{-1}\xi$ satisfies $g \in K$ and $\xi \in \mathfrak{k}$ with $|\xi| < \epsilon$ such that $\tilde{g} \cdot w = w$ then:

$$\phi(g \cdot \exp \xi, w) = \phi(\tilde{g}, w) = \tilde{g} \cdot w = w = \phi(e, w) \text{ and } (\tilde{g}, w) \in U_\epsilon$$

these together with the fact that $\phi|_{U_\epsilon}$ is bi-holomorphic imply that

$$(\tilde{g}, w) \stackrel{G_w}{\sim} (e, w) \in G \times \mathbb{P}W$$

i.e. there is a $h \in G_{z_0}$ such that $(\tilde{g}h^{-1}, hw) = (e, w)$, hence $\tilde{g} = h \in G_{z_0} \cap G_w$. In conclusion, we obtain the following:

Lemma 9.8 (Local Rigidity). *Let $w \in V \subset \mathbb{P}W$ (defined in (68)) and suppose $\tilde{g} \in G_w$ is of the form $\tilde{g} = g \cdot \exp \xi$ with $g \in K$ and $\xi \in \mathfrak{g}$ satisfies $|\xi| < \epsilon$. Then $\tilde{g} \in G_{z_0}$.*

Theorem 9.9. *Let K be a compact Lie group acting on \mathbb{P}^M via a representation $K \rightarrow \mathrm{U}(M+1)$ and $G = K^\mathbb{C}$ be its complexification. Let $z_0 \in \mathbb{P}^N$ with its stabilizer G_{z_0} satisfying $G_{z_0} = (G_{z_0} \cap K)^\mathbb{C}$ and $z_0 \in \Sigma \subset \mathbb{P}^M$ satisfying Condition 9.6. Then there is an G_{z_0} -invariant Zariski open neighborhood $z_0 \in U^{\mathrm{sp}} \subset \mathbb{P}W$ such that for $\forall w \in U^{\mathrm{sp}} \cap G \cdot \Sigma$ we have $G_w < G_{z_0}$.*

Proof. We will first prove it for an *analytic* neighborhood, then by the constructibility we can pass it to a Zariski open neighborhood.

Suppose Assumption 9.6 holds, then the continuity of the slice Σ implies that there is a sufficiently small analytic K_{z_0} -invariant neighborhood $z_0 \in \tilde{V} \subset V \subset \mathbb{P}W$ such that for any $w \in \tilde{V}$, there is a $\xi \in (\mathfrak{k}_{z_0}^\perp)^\mathbb{C}$ satisfying $|\xi| < \delta < \epsilon$ and $z \in \Sigma$ such that $w = \exp \xi \cdot z$. In particular, $\exp \xi \cdot K_z \cdot \exp(-\xi) \subset G_w$ is a maximal compact subgroup of G_w . Since $K_z < K$ is compact we have

$$\begin{aligned} \exp \xi \cdot K_z \cdot \exp(-\xi) &= \{h \cdot \exp(\mathrm{Ad}_{h^{-1}} \xi) \cdot \exp(-\xi) \mid h \in K_z\} \\ &\subset \{g \cdot \exp \sqrt{-1} \zeta \mid \zeta \in \mathfrak{g}, |\zeta| < \epsilon \text{ and } g \in K\}. \end{aligned}$$

By Lemma 9.8, we must have $\exp(-\xi) \cdot K_z \cdot \exp \xi \subset G_{z_0}$. Hence

$$G_{z_0} \supset \left(\exp(-\xi) \cdot K_z \cdot \exp \xi \right)^\mathbb{C} = G_w,$$

since G_{z_0} is reductive. Finally, one notices that the set

$$\{w \in \mathbb{P}W \mid G_w < G_{z_0}\} \supset G_{z_0} \cdot \tilde{V}$$

is G_{z_0} -invariant and constructible. This allows us to choose a G_{z_0} -invariant Zariski open subset $U^{\mathrm{sp}} \supset G_{z_0} \cdot \tilde{V}$, and our proof is completed. \square

Assumption 9.10 (Stabilizer Preserving). There is a G_{z_0} -invariant Zariski open neighborhood of $z_0 \in U^{\mathrm{sp}} \subset \mathbb{P}W$ such that $G_w < G_{z_0}$ for all $w \in U^{\mathrm{sp}}$.

Example 9.11. Notice that Assumption 9.10 does not hold in general, even in the situation that a 1-PS $\alpha(t)$ degenerating $\lim_{t \rightarrow 0} \alpha(t) \cdot z = z_0$, we cannot conclude that $G_{z_t} < G_{z_0}$. Consider the $\mathrm{SL}(2)$ -action on $\mathbb{P}(\mathrm{Sym}^{\otimes 3} \mathbb{C}^2)$ as in Example 9.2.1. The 1-PS

$$\alpha(t) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} t+1/t & -t+1/t \\ -t+1/t & t+1/t \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

degenerates $p(X, Y)$ to $p_0(X, Y) \in \mathbb{P}(\mathrm{Sym}^{\otimes 3} \mathbb{C}^2)$. Then $\mathbb{Z}/3\mathbb{Z} \cong \mathrm{SL}(2)_{p_t} \not\subset \mathrm{SL}(2)_{p_0} = \langle \alpha(t) \rangle \cong \mathbb{G}_m$, and the map

$$\mathrm{SL}(2) \times_{\mathbb{G}_m} \mathbb{P}W \longrightarrow \mathrm{SL}(2) \cdot \mathbb{P}W$$

is *not* finite.

Twisting the linearization of G_{z_0} on $\mathcal{O}_{\mathbb{P}^M}(1)|_{\mathbb{P}W}$ by the inverse of the character corresponding to the action $G_{z_0} \curvearrowright \mathcal{O}_{\mathbb{P}^M}(1)|_{z_0}$ as in the proof of Lemma 3.1, we obtain that $z_0 \in \mathbb{P}W$ is GIT-polystable with respect to the new G_{z_0} -linearization on $\mathcal{O}_{\mathbb{P}^M}(1)|_{\mathbb{P}W}$. Let $U^{\mathrm{ss}} \subset \mathbb{P}W$ denote the GIT-semistable points with respect to this linearization and

$$(69) \quad \pi_W : \mathbb{P}W \supset U^{\mathrm{ss}} \longrightarrow \mathcal{M} := \mathbb{P}W // G_{z_0} \text{ with } \pi_W(z_0) = 0 \in \mathcal{M}$$

denote the GIT quotient map. Let $0 \in B_{\mathcal{M}}(0, r) \subset \mathcal{M}$ be the *open* ball of radius r with respect to a prefixed continuous metric. Then for each $r > 0$, we introduce

Definition 9.12. Let U_r be the *connected component* of

$$(G \cdot \pi_W^{-1}(B(0, r))) \cap \mathbb{P}W \subset U^{\mathrm{ss}}$$

containing z_0 . In particular, U_r is G_{z_0} -invariant.

Let $[\cdot] : G \rightarrow G/G_{z_0}$ denote the quotient map. We say a sequence $\{h_i\} \subset G$ is *bounded* in G/G_{z_0} if and only if $\{\psi^{-1}([h_i])\}$ is contained in a *bounded* subset of $K \times_{K_{z_0}} (\sqrt{-1}\mathfrak{k}_{z_0}^\perp)$, where ψ is the Cartan decomposition (cf. [Sja95, equation (1.8)])

$$(70) \quad \begin{array}{ccc} \psi : K \times_{K_{z_0}} (\sqrt{-1}\mathfrak{k}_{z_0}^\perp) & \longrightarrow & G/G_{z_0} \\ (g, \sqrt{-1}\xi) & \longmapsto & (g \cdot \exp \sqrt{-1}\xi) \cdot G_{z_0} \end{array},$$

which is a K -equivariant *diffeomorphism*.

Assumption 9.13 (Finite Distance). An analytic *open neighborhood* of $z_0 \in U^{\text{fd}} \subset \mathbb{P}W$ is of *finite distance* if there is a *bounded* (in the sense above) set $G_{U^{\text{fd}}} \subseteq G/G_0$ depending only on U^{fd} and z_0 such that for any pair $(z, g) \in U^{\text{fd}} \times G$ satisfying $g \cdot z \in U^{\text{fd}}$, there is an $h \in G$, $[h] \in G_{U^{\text{fd}}} \subseteq G/G_{z_0}$ such that $g \cdot z = h \cdot z$, where $[\cdot] : G \rightarrow G/G_{z_0}$ is the quotient map. It follows from the definition that U^{fd} is G_{z_0} -invariant.

Lemma 9.14. *Suppose both Assumption 9.10 and 9.13 are satisfied. Then there is a positive $\epsilon > 0$ such that for any $0 < r < \epsilon$, U_r satisfies the following: for any sequence $\{(g_i, y_i)\} \in G \times_{G_{z_0}} U_r$ satisfying $z_i = g_i \cdot y_i \rightarrow z_\infty$, as $i \rightarrow \infty$, after passing to a subsequence, there is a*

$$(g_\infty, y_\infty) \in \overline{\{(g_i, y_i)\}_i} \subset G \times_{G_{z_0}} U_r \text{ such that } g_\infty \cdot y_\infty = z_\infty.$$

Proof. First, we notice that after translating z_∞ by a $g \in G$ if necessary, we may assume that $z_\infty \in U_r$. Since we can always pass to a subsequence, we may and will assume $y_i \xrightarrow{i \rightarrow \infty} y_\infty \in U_r$ after a possible decreasing of r .

By Assumption 9.13 that there is a sequence $\{h_i\} \subset G$, with $\{[h_i]\}$ bounded in G/G_0 and satisfies $g_i \cdot y_i = h_i \cdot y_i$, hence $h_i^{-1} \cdot g_i \in G_{y_i}$, $\forall i$. Now by Assumption 9.10, we have

$$h_i^{-1} \cdot g_i \in G_{y_i} < G_{z_0}, \quad \forall i,$$

from which we conclude that $\{[g_i]\}$ is *bounded* in G/G_{z_0} and hence the set $\{(g_i, y_i)\} \subset G \times_{G_{z_0}} U_r$ is precompact. \square

REFERENCES

- [AFS16] Jarod Alper, Maksym Fedorchuk, and David Ishii Smyth, *Existence of good moduli spaces with applications to the log minimal model program for $\overline{M}_{g,n}$* , to appear in Compos. Math. (2016).
- [Alp13] Jarod Alper, *Good moduli spaces for Artin stacks.*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 6, 2349–2042.
- [Art70] Michael Artin, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. (2) **91** (1970), 88–135.
- [Aub78] Thierry Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes.*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95.
- [BBE⁺11] Robert J. Berman, Sébastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zariahi, *Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties*, arXiv:1111.7158 (2011).
- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.
- [Ber15] Bo Berndtsson, *A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry*, Invent. Math. **200** (2015), no. 1, 149–200.
- [Ber16] Robert J. Berman, *K-polystability of \mathbb{Q} -Fano varieties admitting Kähler-Einstein metrics*, Invent. Math. **203** (2016), no. 3, 973–1025.
- [BG14] Robert J. Berman and Henri Guenancia, *Kähler-Einstein metrics on stable varieties and log canonical pairs*, Geom. Funct. Anal. **24** (2014), no. 6, 1683–1730.
- [CDS15a] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.
- [CDS15b] ———, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π* , J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.
- [CDS15c] ———, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.
- [CS14] Xiuxiong Chen and Song Sun, *Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics.*, Ann. of Math. (2) **180** (2014), no. 2, 407–454.
- [Don01] Simon K. Donaldson, *Scalar curvature and projective embeddings. I.*, J. Differential Geom. **59** (2001), no. 3, 479–522.
- [Don02] ———, *Scalar curvature and stability of toric varieties.*, J. Differential Geom. **62** (2002), 289–349.
- [Don08] ———, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry.* **7** (2008), 29–75. Adv. Lect. Math. (ALM), Handbook of geometric analysis. No. 1, Int. Press, Somerville, MA.
- [Don09] ———, *Discussion of the Kähler-Einstein problem.* (2009).
- [Don12a] ———, *Kähler metrics with cone singularities along a divisor*, Essays in mathematics and its applications, Springer, Heidelberg. (2012), 49–79.

- [Don12b] ———, *Stability, birational transformations and the Kähler-Einstein problem*, Surv. Differ. Geom. **17** (2012), 203–228.
- [Don15] ———, *Algebraic families of constant scalar curvature Kähler metrics*, Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations, 2015, pp. 111–137.
- [DS14] Simon K. Donaldson and Song Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math. **213** (2014), no. 1, 63–106.
- [FR06] Joel Fine and Julius Ross, *A note on positivity of the CM line bundle.*, Int. Math. Res. Not., Art. ID 95875, (2006), 14 pp.
- [Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [HMX14] Christopher Hacon, James McKernan, and Chenyang Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), no. 2, 523–571.
- [HX11] Christopher Hacon and Chenyang Xu, *On Finiteness of \mathbf{B} -representations and Semi-log Canonical Abundance*, Accepted by Minimal models and extremal rays-Proceedings of the conference in honor of Shigefumi Mori's 60th birthday, Advanced Studies in Pure Mathematics (2011).
- [Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996.
- [Kol13] ———, *Moduli of varieties of general type*, Handbook of moduli. Vol. II, 2013, pp. 131–157.
- [KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original.
- [KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no. 3, 765–779.
- [KSB88] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338.
- [LS14] Chi Li and Song Sun, *Conical Kähler-Einstein metrics revisited*, Comm. Math. Phys. **331** (2014), no. 3, 927–973.
- [LX14] Chi Li and Chenyang Xu, *Special test configurations and K-stability of Fano varieties*, Annals of Math. **180** (2014), no. 1, 197–232.
- [LWX14] Chi Li, Xiaowei Wang, and Chenyang Xu, *Degeneration of Fano Kähler-Einstein manifolds*, ArXiv:1411.0761 v1 (2014).
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 34, Springer-Verlag, Berlin, 1994.
- [Oda12] Y. Odaka, *The Calabi Conjecture and K-stability*, Int. Math. Res. Not. **10** (2012), 2272–2288.
- [Oda13] Yuji Odaka, *The GIT stability of polarized varieties via discrepancy*, Annals of Math. **177** (2013), no. 2, 171–185.
- [Oda12] ———, *On the moduli of Kähler-Einstein Fano manifolds*, Proceeding of Kinoshita algebraic geometry symposium 2013, arXiv:1211.4833. (2012).
- [Oda15] ———, *Compact moduli space of Kähler-Einstein Fano varieties*, Publ. Res. Inst. Math. Sci. **51** (2015), no. 3, 549–565.
- [OSS16] Yuji Odaka, Cristiano Spotti, and Song Sun, *Compact moduli spaces of Del Pezzo surfaces and Kähler-Einstein metrics*, J. Differential Geom. **102** (2016), no. 1, 127–172.
- [OSY12] Hajime Ono, Yuji Sano, and Naoto Yotsutani, *An example of an asymptotically Chow unstable manifold with constant scalar curvature*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 4, 1265–1287.
- [Pau12] Sean Paul, *CM stability of projective varieties*, arXiv:1206.4923 (2012).
- [PT06] Sean Paul and Gang Tian, *CM stability and the generalized Futaki invariant I.*, arXiv:math/0605278 (2006).
- [PS10] D. H. Phong and Jacob Sturm, *Lectures on stability and constant scalar curvature*, Handbook of geometric analysis, Adv. Lect. Math. (ALM), vol. 14, Int. Press, 2010.
- [Sja95] Reyner Sjamaar, *Holomorphic Slices, Symplectic Reduction and Multiplicities of Representations*, Ann. of Math. (2) **131** (1995), no. 1, 87–129.
- [Spo12] Cristiano Spotti, *Degenerations of Kähler-Einstein Fano Manifolds*, Ph.D. Thesis: arXiv:1211.5334 (2012).
- [SSY14] Song Sun, Cristiano Spotti, and Chengjian Yao, *Existence and deformations of Kähler-Einstein metrics on smoothable \mathbb{Q} -Fano varieties*, To appear in Duke Math. J., arXiv:1411.1725 (2014).
- [SW12] Jian Song and Xiaowei Wang, *The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality*, arXiv:1207.4839 (2012).
- [Sze10] Gabor Székelyhidi, *The Kähler-Ricci flow and K-polystability*, Amer. J. Math. **132** (2010), no. 4, 1077–1090.
- [Tia90] Gang Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class.*, Invent. Math. **101** (1990), no. 1, 101–172.
- [Tia97] ———, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 1–39.
- [Tia12] ———, *Existence of Einstein metrics on Fano manifolds*, Metric and Differential Geometry, **297** (2012), 119–159.
- [Tia13] ———, *Partial C^0 -estimate for Kähler-Einstein metrics*, Commun. Math. Stat. **1** (2013), no. 2, 105–113.
- [Tia15] ———, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156.
- [Tho06] Richard Thomas, *Notes on GIT and symplectic reduction for bundles and varieties*, Surveys in Differential Geometry, Int. Press. **10** (2006), 221–273.
- [WX14] Xiaowei Wang and Chenyang Xu, *Nonexistence of asymptotic GIT compactification*, Duke Math. J. **163** (2014), 2217–2241.
- [Vie95] Eckart Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995.

[Yau78] Shing-Tung Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339-441.

MATHEMATICS DEPARTMENT, STONY BROOK UNIVERSITY, STONY BROOK NY, 11794-3651, USA

E-mail address: `chi.li@stonybrook.edu`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, RUTGERS UNIVERSITY, NEWARK NJ 07102-1222, USA

E-mail address: `xiaowwan@rutgers.edu`

BEIJING INTERNATIONAL CENTER OF MATHEMATICS RESEARCH, 5 YIHEYUAN ROAD, HAIJIAN DISTRICT, BEIJING, 100871, CHINA

E-mail address: `cyxu@math.pku.edu.cn`